PRISM: A LEARNING NOTE

KANAU SHIMADA

Abstract. In this note, we learn the notion of prism, which is introduced by B. Bhatt and P. Scholze in [BS22]. This note contains no original contribution. Any errors or inaccuracies, however, are entirely the responsibility of the author. All rings are assumed to be under $\mathbb{Z}_{(p)}$.

Contents

1. δ-rings

1.1. Definitions.

Definition 1.1. A δ-ring A is a pair (A, δ) of a ring A and a map $\delta : A \to A$ of sets satisfying the following conditions:

- $\delta(1) = \delta(0) = 0$,
- $\delta(x + y) = \delta(x) + \delta(y) - P(x, y)$,
- $\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)$, where $P(x, y) := \frac{(X+Y)^p - X^p - Y^p}{p} \in \mathbb{Z}[X, Y]$.

Let (A, δ_A) and (B, δ_B) be δ-rings. A map $A \to B$ of δ rings is a ring map $f : A \to B$ such that $f \circ \delta_A = \delta_B \circ f$.

We denote by Ring_δ the category of δ-rings.

Remark 1.2. In the above notation, we call the map δ a delta structure on A. These structures are equivalent to ring maps $A \to W_2(A)$ which is a section of $\mathrm{pr}_1 : \mathrm{W}_2(\mathrm{A}) \to A$, where $W_2(A)$ is the ring defined by the following conditions:

- as a set, $W_2(A) = A \times A$,
- $(a, x) + (b, y) = (a + b, x + y - P(a, b))$,

- $(a, x)(b, y) = (ab, a^p y + b^p x + pxy)$.

Definition 1.3. Let A be a ring. A Frobenius lift on A is a ring endomorphism $\phi : A \to A$ such that $\overline{\phi} : A/p \to A/p$ is a Frobenius map.

Lemma 1.4. *Let A be a p-torsionfree ring. Then the following two sets are canonically isomorphic:*

(1) *the set of δ-structures on A,*
(2) *the set of Frobenius lifts on A.*

Proof. Let δ be a delta structure on A, then the map

$$\phi : A \to A : x \mapsto x^p + p\delta(x)$$

is a Frobenius lift on A.

Conversely, let ϕ be a Frobenius lift on A. Then the map

$$\delta : A \to A : x \mapsto \frac{\phi(x) - x^p}{p},$$

which is well defined by the assumption on A, is a delta structure on A. □

To remove the assumption of p-torsionfreeness in the above lemma, we have to go to the derived world:

Remark 1.5. Let A be a p-torsionfree ring. Then the following is a pull-back square in the category of rings:

$$\begin{CD} W_2(A) @>{\mathrm{gh}_1 : (x,y) \mapsto x^p + py}>> A \\ @V{\mathrm{pr}}VV @VV{\mathrm{pr}}V \\ A @>>{\phi \circ \mathrm{pr}}> A/p \end{CD}.$$

Let $P_\bullet \to A$ be a standard simplicial resolution of a (not necessarily p-torsionfree) ring A. Then there exists a pull-back square:

$$\begin{CD} W_2(P_\bullet) @>{\mathrm{gh}_1 : (x,y) \mapsto x^p + py}>> P_\bullet \\ @V{\mathrm{pr}}VV @VV{\mathrm{pr}}V \\ P_\bullet @>>{\phi \circ \mathrm{pr}}> P_\bullet/p \end{CD}.$$

Since all objects in this diagram are fibrant, and the right vertical map is a Kan fibration (see [SP. tag 08NZ, 08P0]), we see that this diagram is also a homotopy pull-back square. Furthermore, we see that the canonical maps $W_2(P_\bullet) \to W_2(A)$ and $P_\bullet/p \to A \otimes^L_{\mathbb{Z}} \mathbb{Z}/p$ are weakly equivalences (because the former one can be seen as a canonical map $P_\bullet \times P_\bullet \to R \times R$ as a map of simplicial sets). Therefore, we have the following pull-back square in the ∞-category of animated rings:

$$\begin{CD} W_2(A) @>{\mathrm{gh}_1 : (x,y) \mapsto x^p + py}>> A \\ @V{\mathrm{pr}}VV @VV{\mathrm{pr}}V \\ A @>>{\phi \circ \mathrm{pr}}> A \otimes^L_{\mathbb{Z}} \mathbb{Z}/p \end{CD}.$$

By Remark 1.2 and this argument, we see that giving a δ-structure on the ring A is equivalent to giving a connected component of the space of the following commutative diagrams of animated rings:

$$\begin{CD} A @>{\Phi}>> A \\ @V{\mathrm{pr}}VV @VV{\mathrm{pr}}V \\ A \otimes^L_{\mathbb{Z}} \mathbb{Z}/p @>>{\phi}> A \otimes^L_{\mathbb{Z}} \mathbb{Z}/p \end{CD}.$$

1.2. Categorical properties.

Theorem 1.6. *The category* Ring_δ *is a presentable category and the forgetful functor* $U : \mathrm{Ring}_\delta \to \mathrm{Ring}$ *preserves all limits and colimits. In particular, this functor has a right adjoint and a left adjoint.*

Proof. (Existence of small limits). Let $\{(A_i, \delta_i)\}_{i \in I}$ be an arbitrary small diagram of δ-rings. Then the pair $(\lim_i A_i, \lim_i \delta_i)$, where each limit is taken in Set, is a limit of this diagram in Ring_δ.

(Existence of small colimits). Let $\{(A_i, \delta_i)\}_{i \in I}$ be an arbitrary small diagram of δ-rings. Warning: the above argument does not holds for colimits, because colimits in Set does not coincides with colimits in Ring. Let $A := \mathrm{colim}_{i \in I} A_i$ be a colimit in Ring. we will define a delta structure on A so that the pair (A, δ) is a colimit of $\{(A_i, \delta_i)\}_{i \in I}$. We define this the one corresponding to the following composition under 1.2

$$A = \mathrm{colim}_i A_i \xrightarrow{\mathrm{colim}_i s_i} \mathrm{colim}_{\mathrm{i}} W_2(A_i) \to W_2(\mathrm{colim}_i A_i) = W_2(A),$$

where s_i is the section corresponding to the given delta structure on A_i.

(The Forgetful functor commutes with limits and colimits). This follows form the above arguments.

(Presentability). It suffices to show that the category Ring_δ is generated by compact objects under filtered colimits. To do so, we determine the compact objects of Ring_δ. Let $\mathbb{Z}\{x\}$ be the δ-ring whose underlying ring is $\mathbb{Z}[x_0, x_1, \dots,]$ and its δ-structure is the one determined by $\delta(x_i) := x_{i+1}$ for any $i \geq 0$. Then there is a natural bijection

$$\mathrm{Hom}_{\mathrm{Ring}_\delta}(\mathbb{Z}\{x\}, A) \cong \mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}[x], A)$$

for any δ-ring A. The same is true for multi-variables. Furthermore, let I be a finitely generated ideal of $\mathbb{Z}[x_1, \dots, x_r]$. Then the ring $\mathbb{Z}\{x_1, \dots, x_r\}/\langle \cup_{n \geq 0} \delta^n(I) \rangle$ has a canonical delta structure coming from the one on $\mathbb{Z}\{x_1, \dots, x_r\}$. One can easily check that there is a canonical bijection:

$$\mathrm{Hom}_{\mathrm{Ring}_\delta}(\mathbb{Z}\{x_1, \dots, x_r\}/\langle \cup_{n \geq 0} \delta^n(I) \rangle, A) \cong \mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}[x_1, \dots, x_r]/I, A)$$

for any δ-ring A. This implies that these δ-rings are compact in Ring_δ. It is also straightforward to see that these objects generates Ring_δ under filtered colimits.

(Existence of adjoint functors). This follows from the above arguments and the adjoint functor theorem. □

Definition 1.7. We denote by W the right adjoint functor to the forgetful functor $\mathrm{Ring}_\delta \to \mathrm{Ring}$.

We denote by Free_δ the left adjoint functor to the forgetful functor $\mathrm{Ring}_\delta \to \mathrm{Ring}$.

Definition 1.8. Let R be a ring. The ring of Witt vectors (a.k.a. Witt ring) of R is the underlying ring of the δ-ring $W(R)$. We denote it by the same symbol $W(R)$.

Remark 1.9. By the proof of Theorem 1.6, we have

$$\mathrm{Free}_\delta(\mathbb{Z}[x_1, \dots, x_r]/I) \cong \mathbb{Z}\{x_1, \dots, x_r\}/\langle \cup_{n \geq 0} \delta^n(I) \rangle.$$

Next, we study the functor W.

Remark 1.10 (y-coordinate)**.** Let R be a ring. Then there is a canonical bijections of sets:

$$\begin{aligned} W(R) &\cong \mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}[T], W(R)) \\ &\cong \mathrm{Hom}_{\mathrm{Ring}_\delta}(\mathbb{Z}\{T\}, W(R)) \\ &\cong \mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}\{T\}, R) \\ &\cong \mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}[T_0, T_1, \dots], R) \\ &\cong R^{\mathbb{N}}. \end{aligned}$$

We denote this composition by $y : W(R) \to R^{\mathbb{N}}$ and call it the y-coordinate of $W(R)$. By definition, the composition

$$y_n : W(R) \xrightarrow{y} R^{\mathbb{N}} \xrightarrow{\mathrm{pr}_n} R$$

coincides with the following composition:

$$W(R) \xrightarrow{\delta^n} W(R) \xrightarrow{\text{counit}} R.$$

The bijection y is not compatible with the ring structures when $R^{\mathbb{N}}$ is the product ring of countably many copies of R. Indeed, $W(R)$ is not isomorphic to $R^{\mathbb{N}}$ as a ring in general. However, we can construct a natural map $W(R) \to R^{\mathbb{N}}$ of rings, which is injective when p is regular on R and isomorphic when p is invertible on R:

Definition 1.11. Let R be a ring. The n-th ghost coordinate of $W(R)$ is the ring map defined by the following composition:

$$\mathrm{gh}_n : W(R) \xrightarrow{\Phi^n} W(R) \xrightarrow{\text{counit}} R.$$

We Denote by

$$\mathrm{gh} : W(R) \to R^{\mathbb{N}}$$

the ring map induced by gh_n for all $n \geq 0$.

We want to fined a explicit description of ghost coordinates. To do so, we have to introduce another coordinate, called x-coordinate, which is compatible with Frobenius lift Φ on $W(R)$.

Lemma 1.12. *There exit polynomials $S_n \in T_n + (T_1, \dots, T_{n-1})\mathbb{Z}[T_0, \dots, T_{n-1}]$ for all $n \geq 0$ such that*

- $S_0 = T_0$ *and* $S_1 = T_1$,
- $\Phi^n(S_0) = S_0^{p^n} + pS_1^{p^{n-1}} + \dots + p^n S_n$ $(=: W_n(S_0, \dots, S_n))$ *for all* $n \geq 0$.

Proof. We will prove this by the induction on $n \geq 0$. The case of $n = 0$ is obvious. Assume that we have already proven the case of $< n$. We will prove the case of n. We can compute as:

$$\begin{aligned} \Phi^n(S_0) &= \Phi(S_0^{p^{n-1}} + pS_1^{p^{n-2}} + \dots + p^{n-1}S_{n-1}) \\ &= \Phi(S_0)^{p^{n-1}} + p\Phi(S_1)^{p^{n-2}} + \dots + p^{n-1}\Phi(S_{n-1}) \\ &= (S_0^p + p\delta(S_0))^{p^{n-1}} + p(S_1^p + p\delta(S_1))^{p^{n-2}} + \dots + p^{n-1}(S_{n-1}^p + p\delta(S_{n-1})) \\ &= (S_0^{p^n} + pS_1^{p^{n-1}} + \dots + p^{n-1}S_{n-1}^p) + p^n(\delta(S_{n-1}) + C), \end{aligned}$$

for some polynomial $C \in (T_1, \dots, T_{n-1})\mathbb{Z}[T_0, \dots, T_{n-1}]$. Define $S_n := \delta(S_{n-1}) + C$, then this polynomial satisfies the second condition of the lemma. Thus it remains to prove that $\delta(S_{n-1}) \in T_n + (T_1, \dots, T_{n-1})\mathbb{Z}[T_0, \dots, T_{n-1}]$. Since $S_{n-1} - T_{n-1} \in (T_1, \dots, T_{n-2})\mathbb{Z}[T_0, \dots, T_{n-2}]$, we have

$$\begin{aligned} \delta(S_{n-1}) &= \delta(T_{n-1} + (S_{n-1} - T_{n-1})) \\ &= \delta(T_{n-1}) + \delta(S_{n-1} - T_{n-1}) + D, \end{aligned}$$

using some polynomial $D \in (T_1, \dots, T_{n-1})\mathbb{Z}[T_0, \dots, T_{n-1}]$. Since

$$\begin{cases} \delta(T_{n-1}) = T_n \\ \delta(S_{n-1} - T_{n-1}) \quad \in (T_1, \dots, T_{n-1})\mathbb{Z}[T_0, \dots, T_{n-1}] \end{cases}$$

hold, we get the conclusion. ∎

From this lemma, we have $\mathbb{Z}[T_0, T_1, \dots] = \mathbb{Z}[S_0, S_1, \dots]$. Thus we have the following canonical bijections for all ring R:

$$\begin{aligned} W(R) &\cong \mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}[T], E(R)) \\ &\cong \mathrm{Hom}_{\mathrm{Ring}_\delta}(\mathbb{Z}\{T\}, E(R)) \\ &\cong \mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}\{T\}, R) \\ &\cong \mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}[T_0, T_1, \dots], R) \\ &= \mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}[S_0, S_1, \dots], R) \\ &\cong R^{\mathbb{N}}. \end{aligned}$$

Using this, we define:

Definition 1.13. Let R be a ring. Then the map $x : W(R) \to R^{\mathbb{N}}$ defined by the above composition is called the x-coordinate of $W(R)$.

Corollary 1.14. *Let $U_n := \sum_{i=0}^n S_i(T_0, \dots, T_n)^{p^{n-i}} p^i \in \mathbb{Z}\{T\}$, where S_i's is the polynomial defined in the above lemma. Then we have $\Phi(U_n) = U_{n+1}$.*

Proof. This follows form the above lemma. ☐

Proposition 1.15. *Let R be a ring. Then the ghost coordinate* $\mathrm{gh} : W(R) \to R^{\mathbb{N}}$ *coincides with the following composition:*

$$W(R) \xrightarrow[\cong]{x} R^{\mathbb{N}} \xrightarrow{(a_n) \mapsto (W_n(a_0, \dots, a_n))} R^{\mathbb{N}},$$

where $W_n(X_0, \dots, X_n) := \sum_{i=0}^n X_i^{p^{n-i}} p^i \in \mathbb{Z}[X_0, \dots, X_n]$.

Proof. Take an arbitrary element a of $W(R)$. Let $f : \mathbb{Z}\{T\} \to W(R)$ be the map of δ-rings defined by $T_0 \mapsto a$. Since f is compatible with the δ-structures, we have

$$\begin{aligned} \Phi^n_{W(A)}(a) &= f(\Phi^n_{\mathbb{Z}\{T\}}(T_0)) \\ &= f(\Phi^n_{\mathbb{Z}\{T\}}(S_0)) \\ &= f(\sum_{i=0}^n S_i^{p^{n-i}} p^i) \\ &= \sum_{i=0}^n x_i(a)^{p^{n-i}} p^i, \end{aligned}$$

where $x_i(a)$ is the i-th x-coordinate of the element $a \in W(R)$. This computation implies the conclusion. ☐

Theorem 1.16. *The functor* $W(-) : \mathrm{Ring} \to \mathrm{Ring}$ *is characterized by the following conditions:*

1. *as a set, there is a natural bijection $W(R) \cong R^{\mathbb{N}}$ for any $R \in \mathrm{Ring}$,*
2. *there is a natural transformation* $\mathrm{gh} : W(-) \to (-)^{\mathbb{N}}$ *of endofunctors on* Ring *such that* $\mathrm{gh}(a_n) = W_n(a_0, \dots, a_n)$ *holds for any $(a_n) \in R^{\mathbb{N}}$ and $n \geq 0$ under the identification of (1).*

Proof. (Existence). The Witt ring functor and the ghost coordinate maps satisfy the desired properties.

(Uniqueness). We have to show that there is at most one natural ring structure on $R^{\mathbb{N}}$ such that the map $\mathrm{gh} : (a_n) \mapsto (W_n(a_0, \dots, a_n))$ is a map of rings. For any ring R, there is a surjective ring map $\widetilde{R} \to R$ form a p-torsionfree ring. Then the induced map $W(\widetilde{R}) \to W(R)$ is also surjective. Thus it suffices to prove the above assertion when R is p-torsionfree. In this case, the conclusion follows form the observation that the map gh is injective. ☐

1.3. **Witt rings.**

In this section, we identify the underlying set of $W(R)$ with $R^{\mathbb{N}}$ under the x-coordinate map.

1.3.1. *Operators on Witt rings.* In this subsection, we define basic operators on rings of Witt vectors and prove some basic properties among them.

Definition 1.17. Let R be a ring. The map

$$[-] : R \to W(R) : a \mapsto (a, 0, 0, \dots)$$

is called the Teichmuller lift map.

Lemma 1.18. *The map*

$$[-] : R \to W(R) : a \mapsto (a, 0, 0, \dots)$$

is a multiplicative (but not additive) section of the canonical projection $W(R) \to R : (a_n) \mapsto a_0$.

Proof. Let R be a ring. We want to prove that

$$(a, 0, \dots)(b, 0, \dots) = (ab, 0, \dots) \in W(R)$$

holds for any elements $a, b \in R$. By considering the map

$$\mathbb{Z}[X, Y] \to R : X \mapsto a, Y \mapsto b,$$

we may assume that R is p-torsionfree. Then the ghost map is injective, so it suffices to show $\mathrm{gh}(a, 0, \dots)\,\mathrm{gh}(b, 0, \dots) = \mathrm{gh}(ab, 0, \dots) \in R^{\mathbb{N}}$. This is obvious. □

Definition 1.19. Let R be a ring. The map

$$V : W(R) \to W(R) : (a_0, a_1, \dots) \mapsto (0, a_0, a_1, \dots)$$

is called the shift map.

Lemma 1.20. *Let R be a ring. Then the shift map $V : W(R) \to W(R)$ is additive (but not multiplicative).*

Proof. Similar to the above proof. □

Definition 1.21. Let R be a ring. The Frobenius lift $F : W(R) \to W(R)$ defined by the δ-structure on $W(R)$ is called the Frobenius map.

Lemma 1.22. *The Frobenius map $F : W(R) \to W(R)$ is characterized by the following properties:*

(1) *F is natural ring map*
(2) *F fits into the following commutative diagram:*

$$\begin{array}{ccc} W(R) & \xrightarrow{\mathrm{gh}} & R^{\mathbb{N}} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle (a_0,a_1,\dots)\mapsto(a_1,a_2,\dots)} \\ W(R) & \xrightarrow[\mathrm{gh}]{} & R^{\mathbb{N}} \end{array} .$$

Proof. By a similar argument in the above lemma, the conclusion follows from Corollary 1.14. □

Lemma 1.23. *Let R be an $\mathbb{F}_p$-algebra. Then the Frobenius map $F : W(R) \to W(R)$ can be written as*

$$(a_0, a_1, \dots) \mapsto (a_0^p, a_1^p, \dots).$$

Proof. It suffices to show that $F((a_n))_n \equiv a_n^p \mod p$ for any $(a_n) \in W(R)$ and any ring R. Since any ring has a surjective ring map from a p-torsionfree ring, we may assume that R is p-torsionfree. In this case, the assertion follows form an elementary computation using ghost coordinates. □

Proposition 1.24. *Let R be a ring. Then the following hold:*

(1) $FV = p$,

(2) $V(xF(y)) = V(x)y$,
(3) $[a](b_n) = (ab_0, a^p b_1, a^{p^2} b_2, \dots)$,

If R is characteristic p, $VF = p$ also holds.

Proof. This follows form a direct computation using ghost coordinates. □

Lemma 1.25. *Let R be a ring. Then the subset $V^n(W(R)) \subseteq W(R)$ is an ideal for any $n \geq 0$.*

Proof. Since V is additive, it remains to show that $V^n W(R) \subseteq W(R)$ is closed under scalar multiplication. This follows form the above proposition. □

Definition 1.26. Let R be a ring. Then the n-th truncated Witt ring $W_n(R)$ is defined by $W_n(R) := W(R)/V^n W(R)$.

Lemma 1.27. *Let R be an $\mathbb{F}_p$-algebra. Then $p^n = V^n(1)$ holds. If moreover R is perfect, then $V^n W(R) = p^n W(R)$ holds.*

Proof. This follows form the above proposition. □

1.3.2. *Witt rings for perfect rings.* In this subsection, we see a relationship between Witt rings and strict p-rings.

Definition 1.28. A ring A is called a strict p-ring if the following conditions hold:
(1) A is p-torsionfree,
(2) A is p-complete,
(3) A/p is perfect.

Example 1.29.
$$\mathbb{Z}_p\langle x_1^{1/p^\infty}, \dots, x_r^{1/p^\infty}\rangle := \mathbb{Z}_p[x_1^{1/p^\infty}, \dots, x_r^{1/p^\infty}]^\wedge_p$$
is a typical example of a strict p-ring.

Strict p-rings are closely related to δ-rings whose Frobenius lifts are bijective.

Definition 1.30. A δ-ring A is called a perfect δ-ring if the Frobenius lift on A is bijective.

Lemma 1.31. *Let A be a perfect δ-ring. Then A is p-torsionfree.*

Proof. Let ϕ be the Frobenius lift on A. Then its derived modulo p is homotopic to the Frobenius map on the animated ring $A/^L p$. Since ϕ is bijective, this Frobenius map is an equivalence. Since the Frobenius map on animated rings kills higher homotopy (see the lemma below), the above argument implies $A[p] = \pi_1(A/^L p) = 0$. □

The following lemma was used in the above proof:

Lemma 1.32 (Frobenius kills higher homotopies)**.** *Let $\phi : A \to A$ be the Frobenius map on an animated $\mathbb{F}_p$-algebra A. Then the induced map $\pi_i(\phi)$ is zero for any $i \geq 1$.*

Proof. Since the Frobenius map coincides with the following composition:
$$A \xrightarrow{\Delta} A^{\times p} \xrightarrow{\text{mult}} A,$$
it suffices to show that the multiplication map
$$A \times A \xrightarrow{m} A$$
induces the zero map on π_i for any $i \geq 1$. Take $x, y \in \pi_i(A)$. Then these two elements define a map $\mathbb{Z}^2[i] \to A$ in $\mathcal{D}_{\geq 0}(\mathbb{Z})$. This defines a map of animated rings $\mathrm{Sym}(\mathbb{Z}^2[i]) \to A$. Since the element $x, y \in \pi_i(A)$ are targeted by the induced map $\pi_i(\mathrm{Sym}(\mathbb{Z}^2[i])) \to \pi_i(A)$, we may assume $A = \mathrm{Sym}(\mathbb{Z}^2[i])$. The element $\pi_i(m)(x, y) \in \pi_i(A)$ corresponds to the composition
$$(\mathbb{Z}^2[i])^{\times 2} \to \mathrm{Sym}(\mathbb{Z}^2[i])^{\times 2} \xrightarrow{m} \mathrm{Sym}(\mathbb{Z}^2[i]),$$
which factors through $\mathbb{Z}^4[2i] \simeq \mathbb{Z}^2[i] \otimes^L_{\mathbb{Z}} \mathbb{Z}^2[i]$. Since we have $\pi_i(\mathbb{Z}^4[2i]) = 0$ for any $i \geq 1$, we get the conclusion. □

Lemma 1.33. *Let R be a perfect $\mathbb{F}_p$-algebra. Then $W(R)$ is p-torsionfree.*

Proof. By Lemma 1.23, the δ-ring $W(R)$ is perfect, so the conclusion follows form Lemma 1.31. □

Lemma 1.34. *Let R be an $\mathbb{F}_p$-algebra. Then the ring $W(R)$ is derived p-complete.*

Proof. By definition, we have $W(R) \cong \lim_n W_n(R) \simeq \mathrm{Rlim}_n W_n(R)$, since transition maps are surjective. Since derived p-complete rings are closed under (derived) limits, it suffices to show that $W_n(R)$ is derived p-complete for any $n \geq 0$. Lemma 1.27 implies that each $W_n(R)$ is p-torsion, so we get the conclusion. □

Lemma 1.35. *The following two categories are equivalent:*

(1) *The category of perfect $\mathbb{F}_p$-algebras.*
(2) *The category of strict p-rings.*

Proof. By Lemma 1.31 and Lemma 1.33, $W(-)$ defines a functor form (1) to (2). Conversely, $(-)/p$ defines a functor from (2) to (1). By Lemma 1.27, there is a canonical equivalence $(-)/p \circ W(-) \simeq \mathrm{id}$. In the following, we will prove that there is a natural isomorphism $W(A/p) \cong A$ for any strict p-ring A. This follows form the deformation theory and $\mathbb{L}_{(A/p)/\mathbb{F}_p} \simeq 0$, which follows from the fact that Frobenius maps kill differential forms. □

Theorem 1.36. *The following three categories are equivalent.*

(1) *The category of perfect $\mathbb{F}_p$-algebras.*
(2) *The category of p-complete perfect δ-rings.*
(3) *The category of strict p-rings.*

Proof. We denote by $\mathscr{C}_i$ the category of (i) in the statement for $i = 1, 2, 3$. Then we have functors:

- $W(-) : \mathscr{C}_1 \to \mathscr{C}_2$,
- $U : \mathscr{C}_2 \to \mathscr{C}_3$,
- $(-)/p : \mathscr{C}_3 \to \mathscr{C}_1$.

We have already proved that the composition $U \circ W(-)$ is an equivalence (Lemma 1.35). This implies that U is essentially surjective. Since U is obviously faithful, the equivalence of $U \circ W(-)$ also implies the full faithfulness of U. □

Lemma 1.37. *Let k be a perfect field of characteristic p. Then $W(k)$ is CDVR.*

Proof. We have already proved that $W(k)$ is p-complete and $W(k)/p = k$. This implies the conclusion. □

Lemma 1.38 (p-adic extension)**.** *Let A be a p-complete perfect δ-ring and x be an element of A. Then x has a unique description of the form*

$$x = \sum_{i=0}^{\infty} [x_i] p^i,$$

where $x_i \in A/p$ and $[-] : A/p \to A$ is the unique multiplicative section of $A \to A/p$.

Proof. Straightforward. □

1.4. Extending δ-structures.

Lemma 1.39. *Let A be a δ-ring, I be an ideal of A such that $\delta(I) \subseteq I$. Then there exists a unique δ-structure on A/I such that the canonical surjection $A \to A/I$ is a map of δ-rings.*

Proof. Straightforward. □

Lemma 1.40. *Let A be a δ-ring. Let S be a multiplicative subset of A with $\phi(S) \subseteq S$, where ϕ is the Frobenius lift defined by the δ-structure of A. Then the localized ring $S^{-1}A$ has a unique δ-structure such that the canonical map $A \to S^{-1}A$ is a map of δ-rings.*

Proof. Straightforward. □

Lemma 1.41. *Let A be a δ-ring. Let I be a finitely generated ideal of A with $p \in I$. Then the classical I-completion $\widehat{A}$ has a unique δ-structure such that the canonical ring map $A \to \widehat{A}$ is a map of δ-rings.*

Proof. We have to extend the map δ on A to the completed ring A. To do this, it suffices to show that the map $\delta : A \to A$ is I-adically continuous. Thus, it reduced to prove the following assertion:

Claim: for all $x \in A$ and all $n \geq 0$, there exists some integer $m \geq 0$ such that $\delta(x+I^m) \subseteq x+I^n$ holds.

By the definition of δ-maps, we have

$$\delta(x + I^m) \subseteq \delta(x) + \delta(I^m) + I^m.$$

Thus, it suffices to show that $\delta(I^m) \subseteq I^n$ holds for sufficiently large $m \geq 0$. Since we have

$$\delta(I^2) \subseteq I + p\delta(I)\delta(I)A \subseteq I,$$

here we used the assumption that I is finitely generated and $p \in I$ holds, we have $\delta(I^{2^{l+1}}) \subseteq I^{2^l}$ for all $l \geq 0$. Therefore, we get the conclusion. □

Lemma 1.42. *Let A be a δ-ring. Let I be a finitely generated ideal of A with $p \in I$. Let B be a I-completely etale A-algebra (that is, B is derived I-complete and the induced map $A/I \to B \otimes^L_A A/I$ is etale). Then the ring B has a unique δ-structure such that the structure map $A \to B$ is a map of δ-rings.*

Proof. By Elkik's theorem, there exists a etale A-algebra B' such that $\widehat{B'} \cong B$ as A-algebras. Let $\psi : A \to W_2(A)$ be the ring map corresponding the δ-structure of A. We see the ring $W_2(B')$ as an A-algebra by the composition $A \xrightarrow{\psi} W_2(A) \to W_2(B')$. Then we have the following commutative diagram:

$$\begin{array}{ccc} A & \longrightarrow & W_2(B) \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}.$$

To define a δ-structure on B, it suffices to take a section $B \to W_2(B)$ fitting into the above commutative diagram. To see this, we first prove that the ring $W_2(B)$ is derived I-complete. Consider the following sequence:

$$0 \to \phi_*B \xrightarrow{b \mapsto (0,b)} W_2(B) \xrightarrow{\mathrm{pr}_1} B \to 0,$$

where ϕ_*B is a A algebra defined by

$$A \xrightarrow{\phi} A \xrightarrow{\mathrm{str}} B$$

and $W_2(B)$ is an A-algebra defined by

$$A \xrightarrow{a \mapsto (a,\delta(a))} W_2(A) \xrightarrow{W_2(\mathrm{str})} W_2(B).$$

Then one can easily check that this sequence is an exact sequence of A-modules. Since B is derived I-complete and $p \in I$, ϕ_*B is also derived I-complete. Since derived I-completeness is closed under extensions, the A-algebra $W_2(B)$ is derived I-complete.

Next, we will prove that the canonical map $W_2(B) \to \lim_n W_2(B)/(0,p)^n$ is surjective. Let $\{(x_n, y_n)\}_n$ be an arbitrary element of the codomain. For each $n \geq 0$, there exists an element $(z, w) \in W_2(B)$ such that $(x_n, y_n) = (x_{n+1}, y_{n+1}) + (0, p)^n(z, w) \in W_2(B)$. This implies that $x_n = x_{n+1} := x \in B$ and $\{y_n\}$ is a Cauchy sequence on B under the p-adic topology. Since B is derived p-complete, there exists an element $y \in B$ such that $y \equiv y_n \mod p^n$ for all $n \geq 0$. Therefore, the element $(x, y) \in W_2(B)$ maps to $\{(x_n, y_n)\}_n$ by the map $W_2(B) \to \lim_n W_2(B)/(0, p)^n$.

Furthermore, if B is also p-adically separated, then one can easily prove that the map $W_2(B) \to \lim_n W_2(B)/(0,p)^n$ is isomorphic. Let $\overline{B} := B/\bigcap_n p^n B$. Since B is derived p-complete, the ring $\overline{B}$ is classically p-complete. Thus the map $W_2(\overline{B}) \to \lim_n W_2(\overline{B})/(0,p)^n$ is isomorphic. Hence, there exists a factorization of the map $W_2(B) \to W_2(\overline{B})$ as follows:

$$W_2(B) \to \lim_n W_2(B)/(0,p)^n \to \lim_n W_2(\overline{B})/(0,p)^n \cong W_2(\overline{B}).$$

Since we have $(\cap_n p^n B)^2 = 0$ (see [Sta26, Tag 0G3G]), this map is a thickening. Therefore, the map $W_2(B) \to \lim_n W_2(B)/(0,p)^n$ is also a thickening. On the other hand, each map $W_2(B)/(0,p)^n \to B$ is also a thickening, because we have $V(B)^3 \subseteq (0,p)W_2(B)$ and $W_2(B)/V(B) = B$. Therefore, we have a lifting solution $B' \to W_2(B)$ of the above diagram. Since $W_2(B)$ is derived I-complete and B is the derived I-completion of B', this map extends to $B \to W_2(B)$, which is the desired section of the canonical surjection $W_2(B) \to B$. $\square$

Another proof. Let $\mathrm{gh}(W_2(B))$ be the image of the ghost coordinate map $W_2(B) \to B \times B$. We first fined a Frobenius lift on B compatible with the one on A. Consider the followng commutative diagram:

$$\begin{array}{ccc} A/I^n & \longrightarrow & \mathrm{gh}(W_2(B \otimes_A^L A/I^n)) \\ \big\downarrow & & \big\downarrow \mathrm{pr}_1 \\ B \otimes_A^L A/I^n & \longrightarrow & B \otimes_A^L A/I^n \end{array}.$$

By the discreteness of $B \otimes_A^L A/I$ and a devissage argument, we see $B \otimes_A^L A/I^n$ is also discrete. Since $A \to B$ is I-completely etale, $A/I^n \to B \otimes_A^L B/I^n$ is etale. From these observations and the fact that pr_1 is a thickening, we get a unique lifting solution $B/I^n \to \mathrm{gh}(W_2(B/I^n))$ in the above diagram. Passing to limit, we get $B \to \mathrm{gh}(W_2(B))$, which defines a Frobenius lift on B compatible with the one on A. We denote it by ϕ_B.

Next, we define a δ-structure on B compatible with the one on A and ϕ_B. To do so, it suffices to fined a unique lifting solution of the following commutative diagram:

$$\begin{array}{ccc} A & \longrightarrow & W_2(B) \\ \big\downarrow & & \big\downarrow \mathrm{gh} \\ B & \longrightarrow & \mathrm{gh}(W_2(B)) \end{array},$$

where the upper horizontal map is defined by the composition

$$A \xrightarrow{a \mapsto (a,\delta(a))} W_2(A) \xrightarrow{W_2(\mathrm{str})} W_2(B)$$

and the lower horizontal map is defined by

$$B \to \mathrm{gh}(W_2(B)) \subseteq B \times B : b \mapsto (b, \phi_B(b)).$$

Since gh is a square-zero extension and $A \to B$ is formally etale, we get the conclusion. $\square$

Remark 1.43. The first proof is due to [BS22], while the second is based on [Rez19]. The author thanks Ryo Suzuki for pointing out the latter reference.

Definition 1.44. Let A be a ring. Let $I \subseteq A$ be an ideal. The Zariski localization of A along $V(I)$ is the filtered colimit $\operatorname{colim} A_f$ where f runs through all elements of A such that $V(I) \subseteq D(f)$. (This is isomorphic to the ring $S^{-1}A$ with $S := A \setminus \cup_{\mathfrak{p} \in V(I)} \mathfrak{p}$).

Lemma 1.45. *Let A be a ring, I be an ideal of it. Let A' be the Zariski localization of A along $V(I)$. Then we have $IA' \subseteq \operatorname{rad} A'$. Furthermore, the ring A' is universal among A-algebras with such property.*

Proof. Recall that $IA' \subseteq \operatorname{rad} A'$ holds if and only if all elements of $1 + IA'$ is invertible. By the definition of Zariski localizations, it suffices to show that all elements of the form $u + x$ for $u \in A^*$ and $x \in I$ are invertible in A'. Let $\mathfrak{p} \in V(I)$ be an arbitrary point. Then we have $u + x \notin \mathfrak{p}$, because $x \in I \subseteq \mathfrak{p}$ and u is invertible. This implies that $V(I) \subseteq D(u+x)$ holds, and this means $u + x$ is invertible in A'.

Next, we will prove the universality of A'. Let $\varphi : A \to B$ be a ring map such that $IB \subseteq \operatorname{rad} B$. Let $f \in A \setminus \cup_{\mathfrak{p} \in V(I)} \mathfrak{p}$ be an arbitrary element. It suffices to show that $\varphi(f)$ is invertible. Since we have $V(I) \subseteq D(f)$, we have $V(IB) \subseteq D(\varphi(f))$. Combining this and $IB \subseteq \operatorname{rad} B$, we have $V(\operatorname{rad} B) \subseteq D(\varphi(f))$. Since the left hand side contains all closed points, we have $D(\varphi(f)) = \operatorname{Spec} B$, that is $\varphi(f) \in B^*$. □

Lemma 1.46. *Let A be a δ-ring. Let I be an ideal of it containing p. Let A' be the Zariski localization of A along $V(I)$. Then there exists a unique δ-structure on A' such that the canonical map $A \to A'$ becomes a map of δ-rings.*

Proof. Write $S := A \setminus \cup_{\mathfrak{p} \in V(I)} \mathfrak{p}$. Let S' be the inverse image of S by $A \to S^{-1}A$. Then we have $S^{-1}A = S'^{-1}A$, so it suffices to show that $\phi(S') \subseteq S'$ holds. Take an arbitrary element $f \in S'$. We have $\phi(f) = f^p + p\delta(f)$. Since $f \in (S^{-1}A)^*$ and $p \in I \subseteq \operatorname{rad}(S^{-1}A)$, we have $\phi(f) \in (S^{-1}A)^*$, so $\phi(f) \in S'$. □

1.5. **Distinguished elements.** In § 1.3.2, we have seen that any perfect $\mathbb{F}_p$-algebras can be written as A/p, where A is a p-complete perfect δ-ring. The element $p \in A$ has a property $\delta(p) \in A^*$. Motivated by this property, we define the following:

Definition 1.47. Let A be a δ-ring. An element $\xi \in A$ is called a distinguished element if $\delta(\xi) \in A^*$ holds.

In the subsequent section, we study rings of the form A/ξ, where A is a (p, ξ)-complete perfect δ-ring and $\xi \in A$ is a distinguished element. In this process, it becomes clear that rings of this form can be regarded as mixed-characteristic analogues of perfect rings. As a consequence, we obtain a powerful tool for extending the theory in positive characteristic to the mixed-characteristic setting. As preparation for this, we study some properties of distinguished elements.

Lemma 1.48. *Let A be a δ-ring, $d \in A$ be a distinguished element, and $u \in A^*$ be an invertible element. Assume that $(p, d) \subseteq \operatorname{rad}(A)$ (for example, A is derived (p, d)-complete). Then the element $ud \in A$ is also distinguished.*

Proof. This follows from the following computations:

$$\begin{aligned} \delta(ud) &= \delta(u)d^p + \delta(d)u^p + p\delta(u)\delta(d) \\ &\equiv \delta(d)u^p \mod \operatorname{rad}(A). \end{aligned}$$

□

Lemma 1.49. *Let A be a δ-ring, $u, d \in A$ be elements. Assume that $(p, d) \subseteq \operatorname{rad}(A)$ holds. If $ud \in A$ is distinguished, then d is also distinguished and u is invertible.*

Proof. This follows from a computation similar to the above proof. □

Lemma 1.50. *Let A be a δ-ring, $d \in A$ be an element. Assume that $(p, d) \subseteq \operatorname{rad}(A)$ holds. Then the following are equivalent:*

(1) *d is distinguished,*
(2) *$p \in (d, \phi(d))$.*

Proof. The implication (1) ⇒ (2) is straightforward, so we will prove the converse direction. Assume that we have $p = ad + b\phi(d)$ for some $a, b \in A$. To see that $\delta(d)$ is a unit, it suffices to show that the ring $A/(p, d, \delta(d))$ is zero (because $(p, d) \subseteq \operatorname{rad}(A)$). Replacing A by its $(p, d, \delta(d))$-completion, we may assume $(p, d, \delta(d)) \subseteq \operatorname{rad}(A)$ holds. The above equation implies

$$d(a + bd^{p-1}) = p(1 - b\delta(d)).$$

Since $\delta(d) \in \mathrm{rad}(A)$ holds, $(1 - b\delta(d))$ is invertible. Since p is distinguished, the right hand side is also distinguished. Since $d \in \mathrm{rad}(A)$, this argument and the above lemma implies that d is also distinguished. This implies the conclusion. □

1.6. **Animated δ-rings.**

Definition 1.51. Denote by

$$W_2^{\mathrm{an}}(-) : \mathrm{Ani}(\mathrm{Ring}) \to \mathrm{Ani}(\mathrm{Ring})$$

the animation of the functor

$$W_2(-) : \mathrm{Ring} \to \mathrm{Ring}\,.$$

Lemma 1.52. *Let R be a ring. Then there is a canonical equivalence $W_2^{\mathrm{an}}(R) \simeq W_2(R)$ of animated rings. (Thus, in the following, we denote $W_2^{\mathrm{an}}(-)$ by $W_2(-)$).*

Proof. This follows form the existence of canonical equivalence $W_2^{\mathrm{an}}(R) \simeq R \times R$ of spaces. □

Definition 1.53. Let A be an animated ring. A δ-structure on A is the map $s : A \to W_2(A)$ of animated rings such that the composition $A \to W_2(A) \xrightarrow{\mathrm{pr}_1} A$ is equivalent to the identity map. The pair (A, s) is called an animated δ-ring.

Remark 1.54. When A is a ring, the notion of δ-structure on A and the notion of animated δ-structure on A is equivalent by Remark 1.2.

2. Prisms

In this section, we introduce prisms and study their basic properties. We then introduce perfect prisms as a special class of prisms, and define perfectoid rings in terms of them.

2.1. **Definitions and first properties.**

Definition 2.1. A prism is a pair (A, I) of a δ-ring A and an ideal $I \subseteq A$ satisfying the following conditions:

1. $I \subseteq A$ is a Cartier divisor (i.e. I is invertible as an A-module),
2. A is derived (p, I)-complete,
3. $p \in (I, \phi(I))$ holds.

A map of prisms $(A, I) \to (B, J)$ is a map $f : A \to B$ of δ-rings such that $f(I) \subseteq J$.

One of the most fundamental property is the rigidity of prismatic structure (see Theorem 2.3). To prove this, we establish the following important lemma, which means that the prismatic structure I locally looks like a distinguished element.

Lemma 2.2. *Let A be a δ-ring, I be an invertible ideal of A such that $(p, I) \subseteq \mathrm{rad}\, A$. Then the following are equivalent:*

1. *$p \in (I, \phi(I))$,*
2. *$p \in (I^p, \phi(I))$,*
3. *there is a ind-Zariski faithfully flat map $A \to A'$ of δ-rings such that IA' is generated by a distinguished element d and $(p, d) \subseteq \mathrm{rad}\, A'$,*
4. *there is a faithfully flat map $A \to A'$ of δ-rings such that IA' is generated by a distinguished element d and $(p, d) \subseteq \mathrm{rad}\, A'$.*
5. *The linearlization $\rho : \phi^* I \to A/I$ of the composition*

$$I \xrightarrow{\delta} A \to A/I$$

is surjective,

Proof. (2) ⇒ (1) and (3) ⇒ (4) are trivial.

(4) ⇒ (2). Since $A \to A'$ is faithfully flat, we may assume $I = (d)$ by replacing A with A'. Then the conclusion follows form $\phi(d) = d^p + p\delta(d)$ and $\delta(d) \in A^*$.

(1) ⇒ (3). Since I is an invertible ideal, there is a Zariski open covering $A \to A''$ with $IA'' = (d)$ for some $d \in A''$. Let A' be the Zariski localization of A'' along (p, d). First, we prove

that the δ-structure on A extends to the one on A'. Let $S \subseteq A$ be the multiplicatively closed subset consisting of elements mapping to invertible elements in A''. Let $S' := \cup_{n\geq 0}\phi^n(S) \subseteq A$. Since $\phi(f) = f^p + p\delta(f)$ and $p \in \operatorname{rad} A'$, A' coincides with the Zariski localization of $S'^{-1}A''$ along (p, d). Since $\phi(S') \subseteq S'$, Lemma 1.40 implies that the δ-structure on A uniquely extends to $S'^{-1}A$ and Lemma 1.46 implies that this δ-structure uniquely extends to A'. By this construction, the map $A \to A'$ is ind-Zariski (in particular, flat) and the map $\operatorname{Spec} A' \to \operatorname{Spec} A$ targets all closed points. Thus the map $A \to A'$ is faithfully flat. Finally, we prove that the element $d \in A'$ is distinguished. By (1), we have $p \in (d, \phi(d))$, so Lemma 1.50 implies the conclusion.

(4) $\Leftrightarrow$ (5). Since (5) can be checked after faithfully flat base change, we may assume $I = (d)$. In this case, the A-linear map in (5) can be identified with

$$A \to A/I : a \mapsto a\delta(d).$$

Therefore (5) is equivalent to the assertion that the element d is distinguished, which is equivalent to (4). $\square$

Theorem 2.3. *Let $f : (A, I) \to (B, J)$ be a map of prisms. Then $I \otimes_A B \cong IB = J$ holds.*

Proof. Since I is an invertible ideal, there is a Zariski open covering $B \to A''$ such that $IA'' = (d)$ for some element $d \in A''$. Then the base change $B \to B \otimes_A A''$ is also a Zariski open covering. Refining this, we can take a Zariski open covering $B \to B''$ such that $JB'' = (e)$ for some element $e \in B''$. By construction, we have the following commutative diagram:

$$\begin{array}{ccc} A & \longrightarrow & A'' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} \\ B & \longrightarrow & B'' \end{array}.$$

Let A' be the Zariski localization of A'' along (p, I) and B' be the Zariski localization of B'' along (p, J). Since $f(I) \subseteq J$ holds, the map f'' induces the canonical map $f' : A' \to B'$. Thus we get the following commutative diagram:

$$\begin{array}{ccc} A & \longrightarrow & A' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ B & \longrightarrow & B' \end{array}.$$

By the proof of the above lemma, A' (resp. B') is a δ-ring with $(p, d) \subseteq \operatorname{rad} A'$ (resp. $(p, e) \subseteq \operatorname{rad} B'$), $A \to A'$ (resp. $B \to B'$) is faithfully flat δ-ring map and $d \in A'$ (resp. $e \in B'$) is distinguished. Thus, to see that the map $I \otimes_A B \to J$ is isomorphic, it suffices to show that the map $I \otimes_A B \otimes_B B' \to J \otimes_B B'$ is so. By the flatness of $A \to A'$ and $B \to B'$, this map can be identified with $dB' \to eB'$. Write $d = ue$ using some element $u \in B'$. Since e and d are distinguished in B', Lemma 1.49 implies $u \in B'^*$. Therefore, we get $I \otimes_A B \cong J$. This implies that the map $IB \to J$ is a surjective B-linear map between locally free B-modules of rank one, so we get the conclusion. $\square$

Definition 2.4. A prism (A, I) is called an orientable prism if I is principal. An orientable prism (A, I) is called an oriented prism if a generator of I is given as a datum.

Remark 2.5. Lemma 2.2 means that any prism is locally looks like an orientable prism. It also implies that any generator of the ideal of an orientable prism is distinguished.

2.2. **Bounded prisms.** For technical reasons, we will mainly focus on a certain class of prisms, called bounded prisms. These prisms are well behaved with respect to homological-algebraic operations such as derived completion, and moreover include almost all prisms that are important in applications.

Definition 2.6. A prism (A, I) is called a bounded prism if A/I has bounded p-power torsion (i.e. $(A/I)[p^\infty] = (A/I)[p^n]$ for some sufficiently large $n \geq 0$).

Lemma 2.7. *Let A be a ring, I be an invertible ideal of A. Then the functor*

$$(-)^\wedge : \mathcal{D}(A) \to \mathcal{D}(A) : M \mapsto \lim_{n\in\mathbb{N}}(M \otimes^L_A A/I^n)$$

is equivalent to the derived I-completion functor.

Proof. Let $f_1, \dots, f_r$ be a generating set of the ideal I. Let $A \to A'$ be a finitely presented faithfully flat map such that $IA' = dA'$ for some non-zerodivisor $d \in A'$. The derived I-completion functor can be written as

$$M \mapsto \lim_{n\in\mathbb{N}}(M \otimes^L_A \mathrm{Kos}(A; f_1^n, \dots, f_r^n)),$$

so it suffices to show that the canonical map

$$\{\mathrm{Kos}(A; f_1^n, \dots, f_r^n)\}_n \to \{A/I^n\}_n$$

is equivalent as pro-objects. It suffices to prove this after the derived base change along $A \to A'$. Indeed, Let K_n be the cofiber of the canonical map

$$\mathrm{Kos}(A; f_1^n, \dots, f_r^n) \to A/I^n$$

, and assume $\{K_n\} \otimes^L_A A'$ is pro-zero. Since the full subcategory $\mathscr{C} \subseteq \mathcal{D}(A)$ of objects $X \in \mathcal{D}(A)$ such that $X \otimes^L_A \{K_n\}$ is pro-zero is a thick $\otimes$-ideal and contains $A' \in \mathcal{D}(A)$, the descendability of the map $A \to A'$ implies $A \in \mathscr{C}$. This means $\{K_n\}(\simeq \{K_n\} \otimes^L_A A)$ is also pro-zero. Therefore, we may assume $I = (d)$ by replacing A with A'. In this case, we have

$$\mathrm{Kos}(A; d^n) \simeq A/d^n (= A/I^n),$$

because d is a non-zerodivisor. Thus it suffices to show that the following two canonical maps are equivalent as maps of pro-objects:

(1) $\{\mathrm{Kos}(A; d^n)\}_n \to \{\mathrm{Kos}(A; d^n, f_1^n, \dots, f_r^n)\}_n$,
(2) $\{\mathrm{Kos}(A; f_1^n, \dots, f_r^n)\}_n \to \{\mathrm{Kos}(A; d^n, f_1^n, \dots, f_r^n)\}_n$.

Since we can prove them in a similar way, we only deal with (2). By [Sta26, Tag 0629], we have

$$\begin{aligned}\{\mathrm{Kos}(A; f_1^n, \dots, f_r^n, d^n)\}_n &\simeq \{\mathrm{Cofib}(\mathrm{Kos}(A; f_1^n, \dots, f_r^n) \xrightarrow{\times d^n} \mathrm{Kos}(A; d^n, f_1^n, \dots, f_r^n))\}_n \\ &\simeq \mathrm{Cofib}(\{\mathrm{Kos}(A; f_1^n, \dots, f_r^n)\}'_n \xrightarrow{\{\times d^n\}} \{\mathrm{Kos}(A; f_1^n, \dots, f_r^n)\}_n),\end{aligned}$$

where the transition map of $\{\mathrm{Kos}(A; f_1^n, \dots, f_r^n)\}'_n$ is additionally multiplied by d to the transition map of $\{\mathrm{Kos}(A; f_1^n, \dots, f_r^n)\}_n$. Then [Sta26, Tag 0663] and $d \in (f_1, \dots, f_r)$ implies that $\{\mathrm{Kos}(A; f_1^n, \dots, f_r^n)\}'_n$ is pro-zero. Thus we get the conclusion. □

Remark 2.8. The above proof is not very difficult, but the author wonders whether there exists a simpler proof.

Definition 2.9. Let A be a ring, I be a finitely generated ideal of A. Then an object $M \in \mathcal{D}(A)$ is said to be I-completely (faithfully) flat if $M \otimes^L_A A/I$ is discrete and (faithfully) flat as a discrete A/I-module.

We will prove some preliminary lemmas concerning this:

Lemma 2.10. *Let A be a ring, $I \subseteq A$ be a finitely generated ideal and $M \in \mathcal{D}(A)$ be an object. Then the following assertions are equivalent:*

- *M is I^n-completely flat over A for any n,*
- *M is I-completely flat over A,*
- *M is I^n-completely flat for some $n \geq 0$.*

Furthermore, if $J \subseteq A$ is a finitely generated ideal such that $J \subseteq \sqrt{I}$ and M is I-completely flat, then M is also J-completely flat.

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is trivial and (3) $\Rightarrow$ (2) follows form the last assertion. First, we will prove (2) $\Rightarrow$ (1). Assume that M is I-completely flat. Consider the following fiber sequence:

$$M \otimes^L_A I^n/I^{n+1} \to M \otimes^L_A A/I^{n+1} \to M \otimes^L_A A/I^n \to .$$

Since $M \otimes^L_A I^n/I^{n+1} \simeq M \otimes^L_A A/I \otimes^L_{A/I} I^n/I^{n+1}$ holds, the inductive argument implies the discreteness of each $M \otimes^L_A A/I^n$. To prove the flatness of $M \otimes^L A/I^n$, take an arbitrary A/I^n-module N. The exact sequence

$$0 \to IN \to N \to N/IN \to 0$$

induces the following fiber sequence:

$$IN \otimes_{A/I^n} (M \otimes^L_A A/I^n) \to N \otimes^L_{A/I^n} (M \otimes^L_A A/I^n) \to N/IN \otimes^L_{A/I^n} (M \otimes^L_A A/I^n) \to .$$

Then the inductive argument implies the discreteness of each $N \otimes^L_{A/I^n} (M \otimes^L_A A/I^n)$.

Next, we will prove the last assertion. Let J be a finitely generated ideal such that $J \subseteq \sqrt{I}$. Assume M is I-completely flat over A. Then $I^n \subseteq J$ folds for sufficiently large $n \geq 0$. By the above argument, we see that M is I^n-completely flat. Thus $M \otimes^L_A A/I^n$ is discrete and flat over A/I^n. Since there is a canonical map $A/I^n \to A/J$, this implies that $M \otimes^L_A A/J$ is also discrete and flat over A/J. □

Lemma 2.11. *Let A be a ring, $f \in A$ be an element. Assume that A has bounded f-power torsion. Then the following assertions are equivalent for any $M \in \mathcal{D}(A)$:*

(1) *M is derived f-complete and f-completely flat,*
(2) *M is discrete, classically f-complete and having bounded f-power torsion, and M/f^n are flat over A/f^n for all $n \geq 0$.*

Furthermore, under the above equivalent conditions, $M[f^n] \cong M \otimes_A A[f^n]$ holds for any $n \geq 0$.

Proof. (1) $\Rightarrow$ (2). Assume that M is f-completely flat and derived f-complete. Since A has bounded f-power torsion, we have

$$M \simeq \lim_n (M \otimes^L_A A/f^n).$$

Since M is f-completely flat over A, $M \otimes^L_A A/f^n$ is discrete for any $n \geq 0$. Indeed, we can prove this by induction. The case of $n = 1$ follows form the assumption. Assume that we have proved the case for $< n$ and will prove the case for n. Consider the following fiber sequence:

$$M \otimes^L_A f^{n-1}A/f^nA \to M \otimes^L_A A/f^n \to M \otimes^L_A A/f^{n-1} \to .$$

We have

$$M \otimes^L_A f^{n-1}A/f^nA \simeq (M \otimes^L_A A/f) \otimes^L_{A/f} f^{n-1}A/f^nA.$$

Since $M \otimes^L_A A/f$ is flat over A/f, this object is concentrated in degree zero. From this observation and the induction hypothesis, we see that the object $M \otimes^L_A A/f^n$ is also concentrated in degree zero. Therefore, $M \simeq \lim_n H_0(M)/f^n$, in particular, M is discrete and classically f-complete. By the above lemma, we also know the flatness of M/f^n over A/f^n. The remaining assertion that M has bounded f-power torsion follows from the last assertion, which will be proved in the final paragraph.

(2) $\Rightarrow$ (1). Since A has bounded f-power torsion, there is a canonical pro-isomorphism

$$\{A/f^n\} \simeq \{A \xrightarrow{f^n} A\}.$$

This implies

$$\{M \otimes^L_A A/f^n\} \simeq \{M \xrightarrow{f^n} M\}.$$

Since M has bounded f-power torsion, we have

$$\{M \xrightarrow{f^n} M\} \simeq \{M/f^n\}.$$

Combining them, we get

$$\{M \otimes^L_A A/f^n\} \simeq \{M/f^n\},$$

which implies

$$\{M \otimes_A^L A/f\} \simeq \{M\} \otimes_{\{A\}}^L \{A/f^n\} \otimes_{\{A/f^n\}}^L \{A/f\} \simeq \{M/f^n\} \otimes_{\{A/f^n\}}^L \{A/f\} \simeq \{M/f\},$$

where we used the flatness of M/f^n over A/f^n in the last equivalence. This implies $M \otimes_A^L A/f \simeq M/f$.

In the end, we prove the last assertion under the assumption of (1). Consider the following fiber sequence:

$$M \otimes_A^L A[f^n] \to \mathrm{Kos}(M; f^n) \to M \otimes_A^L A/f^n \to .$$

By the above lemma, we have $M \otimes_A^L A/f^n \simeq M/f^n$. Thus we have

$$\mathrm{Fib}(\mathrm{Kos}(M; f^n) \to M \otimes_A^L A/f^n) \simeq M[f^n].$$

Combining them, we get $M[f^n] \simeq M \otimes_A A[f^n]$. □

The following lemma is a key result to study homological-algebraic properties of bounded prisms.

Lemma 2.12. *Let A be a ring, I be an invertible ideal of A. Assume that A/I has bounded p-power torsion. Let $M \in \mathcal{D}(A)$ be a (p, I)-completely flat derived (p, I)-complete object. Then the following properties hold:*

1. *M is discrete,*
2. *M is classically (p, I)-complete,*
3. *M is I-power torsionfree (i.e. $M \otimes_A^L A/I^n$ is concentrated in degree zero for any $n \geq 0$),*
4. *M/I^n has bounded p-power torsion for any $n \geq 0$.*

Proof. By Lemma 2.7, we have $M \simeq \lim_n M \otimes_A^L A/I^n$. To apply Lemma 2.11, we have to check that $M \otimes_A^L A/I^n$ is p-completely flat and derived p-complete. The former assertion follows from our assumption. For the latter assertion, we have $M \otimes_A^L A/I^n \simeq (M \otimes_A^L I^n \to M)$, so it suffices to show that the complex $M \otimes_A^L I^n$ is derived p-complete. Since I^n is finite projective, we have an isomorphism $I^n \bigoplus P \simeq A^{\oplus l}$ for some A-module P. So the conclusion follows form the fact that any retraction of derived p-complete object is derived p-complete, which follows form the definition.

Applying Lemma 2.11, we see that $M \otimes_A^L A/I^n$ is discrete and having bounded p-power torsion. Therefore, we have:

$$M \simeq \lim_n M \otimes_A^L A/I^n \simeq \lim_n M/I^n,$$

which is discrete. Since M/I^n has bounded p-power torsion, we have:

$$\begin{aligned} M &\simeq \lim_n \lim_m (M \otimes_A^L A/I^n) \otimes_A^L A/p^m \\ &\simeq \lim_{n,m} M/(I^n, p^m) \\ &\simeq \lim_n M/(p, I)^n, \end{aligned}$$

which means that M is classically (p, I)-complete. □

By the preceding lemmas, we obtain the following corollaries concerning the homological-algebraic properties of bounded prisms.

Corollary 2.13. *Let (A, I) be a bounded prism. Then A is classically (p, I)-complete.*

Proof. This is a special case of Lemma 2.12. □

Corollary 2.14. *Let (A, I) be a bounded prism, $M \in \mathcal{D}(A)$ be a (p, I)-completely flat derived (p, I)-complete object. Then the following properties are satisfied:*

1. *M is discrete,*
2. *M is classically (p, I)-complete,*
3. *$M[I^n] = 0$ for any n,*
4. *M/I^n has bounded p-power torsion for any n.*

Proof. This is a special case of Lemma 2.12. □

Corollary 2.15. *Let (A, I) be a bounded prism, B be a derived (p, I)-complete δ-A-algebra (p, I)-completely flat over B. Then the pair (B, IB) is a bounded prism.*

Proof. This is a special case of Lemma 2.12. □

Corollary 2.16. *Let (A, I) be a bounded prism. Then the category of (p, I)-completely flat bounded prisms under (A, I) is equivalent to the category of derived (p, I)-complete (p, I)-completely flat δ-A-algebras.*

Proof. This follows form Corollary 2.15 and Theorem 2.3. □

Corollary 2.17. *Let (A, I) be a bounded prism. Then there is a (p, I)-completely faithfully flat map $(A, I) \to (B, IB)$ of bounded prisms such that (B, IB) is orientable.*

Proof. This follows from Corollary 2.15 and Lemma 2.2. □

2.3. **Transversal prisms.** In this section, we will study basic properties of transversal prisms. Readers can safely skip this section on a first reading.

Definition 2.18. A prism (A, I) is said to be transversal if A/I is p-torsionfree.

A goal of this subsection is to show that the full subcategory $\mathrm{Prism}^{\pitchfork} \subseteq \mathrm{Prism}$ is cofinal. To see this, we prove Theorem2.22 and Theorem 2.25.

Definition 2.19. A preprism is the pair (A, I) of a δ-ring A and an invertible ideal $I \subseteq A$. A map $(A, I) \to (B, J)$ of preprisms is a δ-ring map $f : A \to B$ such that $f(I)B = J$.

A preprism (A, I) is said to be transversal if A/I is p-torsionfree.

Lemma 2.20. *Let (A, I) be a prism. Then the following assertions are equivalent:*

(1) *(A, I) is transversal,*
(2) *A is p-torsionfree and the natural map $I/pI \to A/p$ is injective.*

Proof. Assume (A, I) is transversal. Then the devissage argument implies that A/I^n is p-torsionfree for any $n \geq 1$. Since A is classically p-complete, A is also p-torsionfree.

Thus we may assume A is p-torsionfree. To see the equivalence of (1) and (2), we may assume $I = dA$ by replacing A with some Zariski cover which trivialize I. In this case, the conclusion follows from the following computation:

$$\begin{aligned} A/(p, d) \oplus ((A/d)[p])[1] &\simeq (A \xrightarrow{d} A) \otimes_A (A \xrightarrow{p} A) \\ &\simeq A/(p, d) \oplus ((A/p)[d])[1]. \end{aligned}$$

□

The following theorem means that transversal preprism has a "prismatification".

Theorem 2.21. *Let (A, I) be a transversal preprism. Then there is a map $(A, I) \to (B, J)$ of preprisms satisfying the following properties:*

- *(B, J) is a transversal prism*
- *For any map $(A, I) \to (C, K)$ of preprisms with (C, K) prism, there is a unique map $(B, J) \to (C, K)$ of prisms under (A, I).*

Proof. Define the subset $U \subseteq \mathrm{SpecA}$ by the following:

$$U := \{x \in \operatorname{Spec} A | \operatorname{Im}(\rho) \otimes_A \kappa(x) \neq 0\},$$

where $\rho : I \to A/I$ is the A-linearlization of the following ϕ-semilinear map:

$$I \xrightarrow{\delta} A \to A/I$$

(c.f. Lemma 2.2). Since $\operatorname{Im} \rho$ is finitely generated, U is open. One can also check that U is affine. Indeed, consider a faithfully flat δ-ring map $A \to A'$ such that $IA' = (d)A'$ using some $d \in A'$. Then

$$U \times_{\mathrm{SpecA}} \operatorname{Spec} A' = \{x \in \operatorname{Spec} A' | \delta(d) \neq 0 \in \kappa(x)\} = D(\delta(d))$$

holds, so the fpqc descent for affine morphisms implies U is affine. Therefore we can write as $U = \operatorname{Spec} B_0$. Write $J_0 := IB_0$. Furthermore, define $B := \lim_{n,m} B_0/(p^n, J_0^m)$ and $J := J_0 B$. We want to check that the ring B has a natural δ-structure and the pair (B, J) is a transversal prism satisfying the suitable universal property.

(B is derived (p, J)-complete). Since $A \to B_0$ is flat, J_0 is an invertible ideal of B_0. Therefore, B is a derived (p, I)-completion of B_0. Furthermore, B is also a classical (p, J)-completion of B_0 (see Lemma 2.12).

(J is an invertible ideal). Consider the following exact sequence:

$$0 \to I \otimes_A B_0/J_0^m \to B_0/J_0^{m+1} \to B_0/J_0 \to 0.$$

Applying the snake lemma to the multiplication-by-p^n map on this exact sequence and recalling $(B_0/J_0)[p] = 0$, we get the following exact sequence:

$$0 \to I \otimes_A B_0/(J_0^m, p^n) \to B_0/(J_0^{m+1}, p^n) \to B_0/(J_0, p^n) \to 0.$$

Taking limits on n and m, we get the following exact sequence:

$$0 \to I \otimes_A B \to B \to B/J \to 0.$$

This implies $J \cong I \otimes_A B$, so J is an invertible ideal.

(δ-structure on B). Since B is p-torsionfree, it suffices to define a Frobenius lift on B compatible with the one on A. Let ϕ_A be the Frobenius lift on A defined by its δ-structure. Since $\phi_A(I) \subseteq I^p + (p)$, we have $\phi_A(I^{2k}) \subseteq I^{pk} + (p^k)$. Thus $\phi_A : A \to A$ induces $A/(I^{2k}, p^k) \to A/(I^{pk}, p^k)$. This induces $B/(J_0^{2k}, p^k) \to B_0/(J_0^{pk}, p^k)$. Passing to limit, we get a ring map $\phi_B : B \to B$, which is a Frobenius lift on B compatible with ϕ_A.

((B, J) is a prism). To see that the preprism (B, J) is a prism, it suffices to show the surjectivity of the map $\rho_B : \phi_B^* J \to B/J$ defined as the B-linearlization of the composition $J \xrightarrow{\delta} B \to B/J$ (see Lemma 2.2). By complete Nakayama's lemma, it suffices to show the surjectivity of $\phi_B^* J \to B/(p, J)$, which follows form the fact that the map $\operatorname{Spec} B/(J, p) \to \operatorname{Spec} A$ factors through $U \subseteq \operatorname{Spec} A$.

(Universality of (B, J)). Let $(A, I) \to (C, K)$ be an arbitrary map of preprisms with (C, K) prism. By Lemma 2.2, the map $\operatorname{Spec} C \to \operatorname{Spec} A$ factors through U. Let $B_0 \to C$ be the corresponding ring map. Since C is derived (p, J_0)-complete (use $K = IC = J_0 C$), it induces $B \to C$. By construction this satisfies $JC = K$. In the end, we will check that this map is compatible with their δ-structures. To see this, it suffices to show the commutativity of the following diagram:

$$\begin{array}{ccc} B & \xrightarrow{b \mapsto (b, \delta(b))} & W_2(B) \\ \downarrow & & \downarrow \\ C & \xrightarrow[c \mapsto (c, \delta(c))]{} & W_2(C) \end{array}.$$

Since $A \to C$ is a map of δ-rings, the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{a \mapsto (a, \delta(a))} & W_2(A) \\ \downarrow & & \downarrow \\ C & \xrightarrow[c \mapsto (c, \delta(c))]{} & W_2(C) \end{array}.$$

By construction, the composition $\operatorname{Spec} W_2(B) \to \operatorname{Spec} B \to \mathrm{SpecA}$ factors through $\operatorname{Spec} B_0 \subseteq \operatorname{Spec} A$. Thus the above commutative diagram implies the following commutative diagram:

$$\begin{array}{ccc} B_0 & \xrightarrow{b \mapsto (b,\delta(b))} & W_2(B) \\ \downarrow & & \downarrow \\ C & \xrightarrow[c \mapsto (c,\delta(c))]{} & W_2(C) \end{array}.$$

Since $W_2(C)$ is (p,I)-complete, this commutative diagram induces the desired commutative diagram. □

Theorem 2.22. *Let (A,I) be a prism. Then there is a map $(B,J) \to (A,I)$ of prisms with (B,J) transversal.*

To prove this, we establish the following two lemmas.

Lemma 2.23. *Let A be a ring, M be a finitely generated projective A-module. Then there is a smooth $\mathbb{Z}$-algebra A_0 and a finitely generated projective A_0-module M_0 such that $M_0 \otimes_{A_0} A \cong M$.*

Proof. We may assume M is a direct summand of A^n. The functor

$$F : \mathrm{Ring} \to \mathrm{Set} : R \mapsto \{f \in \operatorname{End}_R(R^n) | f^2 = f\}$$

can be corepresented by the ring defined by

$$A_0 := \mathbb{Z}[X_{i,j} | i,j \in \{1,2,\dots,n\}]/(\sum_{k=1}^{n} X_{i,k}X_{k,j} - X_{i,j} | i,j \in \{1,2,\dots,n\}),$$

which is smooth over $\mathbb{Z}$ (use Jacobian criterion). Let M_0 be the finite projective A_0-module corresponding to the universal element of $F(A_0)$. Then the pair (A_0, M_0) is the desired one. □

Lemma 2.24. *Let R be a noetherian ring. Assume that R is p-torsionfree and R/p is formally smooth over $\mathbb{F}_p$. Then $A := \mathrm{Free}_\delta(R)^\wedge_p$ is flat over R.*

Proof. Take a surjective ring map $\pi : B := R[x_i | i \in I]^\wedge_p \to \mathrm{Free}_\delta(R)$. Since the domain of this map is flat over R, it suffices to show that this map has a section as a map of R-algebras. Let $\alpha : R \to W(A)$ be the ring map corresponding to $\mathrm{id_A}$ under the following bijections:

$$\operatorname{Hom}(A,A) \cong \operatorname{Hom}_\delta(A, W(A)) \cong \operatorname{Hom}(R, W(A)).$$

By adjunction, it reduces to show that the map $\alpha : R \to W(A)$ has a lift along $W(\pi) : W(B) \to W(A)$. To prove this, we will construct a compatible system of lifts $R \to W_n(B)$ of $R \to W_n(A)$. Since the case for $n = 1$ is trivial, assume that we have prove the case for $\leq n$ and prove the case for $n+1$. To do this, it suffices to show that the map

$$R \to W_{n+1}(A) \times_{W_n(A)} W_n(B)$$

has a lift along the surjection

$$W_{n+1}(B) \to W_{n+1}(A) \times_{W_n(A)} W_n(B).$$

Write $I := \ker \pi$. Then the kernel of the surjection $W_{n+1}(B) \to W_{n+1}(A) \times_{W_n(A)} W_n(B)$ is $V^n(I)$. Furthermore, we have

$$W_{n+1}(B) \cong \lim_m W_{n+1}(B)/V^n(p^m I).$$

Since

$$\ker(W_{n+1}(B)/V^n(p^{m+1}I) \to W_{n+1}(B)/V^n(p^m I))$$

is killed by a p-power, the conclusion follows from a inductive argument using the formal smoothness of $R/^L p$. □

Proof of the theorem. Apply Lemma 2.23 to the pair (A, I), we get a smooth $\mathbb{Z}$-algebra A_0 and a line bundle L_0 on A_0 such that $L_0 \otimes_{A_0} A \cong I$. Let $B_0 := \mathrm{Sym}_{A_0}(L_0)$, $J_0 := \mathrm{Sym}_{A_0}^{>0}(L_0)$. Then we have a map $(B_0, J_0) \to (A, I)$ of pairs such that $J_0 A = I$. Applying Lemma 2.24 to B_0, we see that $\mathrm{Free}_\delta(B_0)_p^\wedge$ is a flat B_0-algebra. Thus the ideal $J_o \mathrm{Free}_\delta(B_0)$ is invertible. Therefore, the induced map

$$(\mathrm{Free}_\delta(B_0)_p^\wedge, J_0 \mathrm{Free}_\delta(B_0)_p^\wedge) \to (A, I)$$

is a map of preprisms and its domain is a transversal preprism. Therefore, the conclusion follows form Theorem 2.21. □

Theorem 2.25. *Let (A, I) be a non-zero transversal prism, (B, J) be a bounded prism. Then there is a coproduct $(A, I) \coprod (B, J)$ in* Prism *and the canonical map $(B, J) \to (A, I) \coprod (B, J)$ is faithfully flat*

Proof. □

Corollary 2.26. *The category* $\mathrm{Prism}^{\pitchfork}$ *of transversal prisms is cofinal in* Prism*.*

Proof. By Quillen's theorem A, it suffices to show that the slice category $\mathrm{Prism}^{\pitchfork}_{/(A,I)}$ is weakly contractible for any prism (A, I). By Theorem 2.22, this category is non-empty. Then theorem 2.25 implies the conclusion. □

2.4. **Perfect prisms and perfectoid rings.**

2.4.1. *Definitions and First properties of perfect rings.*

Definition 2.27. A prism (A, I) is said to be perfect if the Frobenius lift on A is bijective.

First, we will prove that perfect prisms are bounded and orientable. To do so, we show the following preliminary lemmas:

Lemma 2.28. *Let (A, I) be a prism and ϕ be the Frobenius lift defined by the δ-structure on A. Then the following hold:*

(1) *$\phi(I)A \subseteq A$ is a principal ideal generated by a distinguished element,*
(2) *$\phi^* I \cong A$ as A-modules,*
(3) *$I^p \cong A$ as A-modules.*

Proof. (1). By Lemma 2.2, we have $p = a + b$ for $a \in I^p$ and $b \in \phi(I)A$. Since $a \in I^p$,

$$\delta(b) = \delta(-a) + \delta(p) - \frac{(p-a)^p - p^p - (-a)^p}{p}$$

is an invertible element. Thus, it remains to prove that $\phi(I)A$ is generated by b. To see this, we may assume $I = (d)$ for some distinguished element $d \in A$ by Lemma 2.2. In this case, we can write as $\phi(d) = ub$ using some $u \in A$. We want to show that the element u is invertible. To do so, it suffices to show $A/(p, d, u) = 0$. We may assume $(p, d, u) \subseteq \mathrm{rad}\, A$ by replacing A with its Zariski localization along (p, d, u). Since we can write $p = d^p s + \phi(d)u$ using some $a \in A$, we have

$$p(-\delta(d)u) = d(d^{p-1}(1 + s)).$$

By Lemma 1.48, the left hand side term is distinguished. From this and Lemma 1.50, d is invertible, which implies $A/(u, d, p) = 0$.

(2). Consider the canonical surjection

$$\phi^* I = I \otimes_{A,\phi} A \xrightarrow{x \otimes 1 \mapsto \phi(x)} \phi(I)A.$$

Let $x \in \phi^* I$ be a lift of a generator of $\phi(I)A$ along this map. We will prove that the map $A \xrightarrow{1 \mapsto x} \phi^* I$ is isomorphic. To do so, it suffices to show its surjectivity, so we may assume that A is a perfect p-complete δ-ring by replacing A by $(A_{\mathrm{perf}})_p^\wedge := (\mathrm{colim}_\phi A)_p^\wedge$. In this case, the map $\phi^* I \to \phi(I)A$ is an isomorphism, so the map $A \to \phi^* I$ is surjective by our choice of x.

(3). Since $\phi^* I$ is projective and $\phi^* I \otimes_A A/p \cong I^p \otimes_A A/p$, there is a lift $\phi^* I \to I^p$ which is isomorphic by Nakayama's lemma. Thus (3) follows from (2). □

Lemma 2.29. *Let A be a p-torsionfree derived p-complete δ-ring such that A/p is reduced. Let $d \in A$ be a distinguished element. Then d is a non-zerodivisor and $(A/d)[p] = (A/d)[p^2]$ holds.*

Proof. Let $x \in A$ be an element such that $dx = 0$. Then we have

$$0 = \delta(dx) = \delta(d)x^p + \delta(x)\phi(d).$$

Since $\phi(dx) = 0$, this implies $x^p\phi(x) = 0$, because $\delta(d)$ is invertible. In particular, we have $x^{2p} = 0 \in A/p$. Since A/p is reduced, this implies $x = px'$ for some $x' \in A$. Since A is p-torsionfree, we also have $\phi(x')x'^p = 0$. Repeating this, we get $x \in \cap_n p^n A = 0$.

Next, we will prove that $(A/d)[p] = (A/d)[p^2]$. Take an arbitrary element $x \in (A/d)[p^2]$. Then we can write as $p^2x = dy$ using some $y \in A$. We have $\delta(p^2x) = \delta(p(px)) \in pA$. On the other hand, we have $\delta(dy) = y^p\delta(d) + \delta(y)\phi(d)$. Thus we have

$$y^p\delta(d) \equiv -\delta(y)\phi(d) \mod p.$$

Since $\phi(d)\phi(y) = \phi(dy) = \phi(p^2x) \in pA$, this implies

$$y^p\phi(y) \equiv 0 \mod p,$$

so we have $y = py'$ for some $y' \in A$. Since A is p-torsionfree, we get $x \in (A/d)[p]$. □

Corollary 2.30. *Let (A, I) be a perfect δ-ring. Then I is generated by a non-zerodivisor and $(A/I)[p] = (A/I)[p^2]$ holds. In particular, (A, I) is a bounded orientable prism.*

Proof. $(A/I)[p] = (A/I)[p^2]$ follows from the above lemma. We will prove the remaining part. Since the Frobenius lift $\phi : A \to A$ is an isomorphism of rings, it suffices to show that the ideal $\phi(I) \subseteq A$ is principal. This follows from Lemma **??**. □

Corollary 2.31. *Let* Prism *be the category of prisms,* $\mathrm{Prism}^{\mathrm{perf}}$ *be the category of perfect prisms. Then the canonical inclusion* $\mathrm{Prism}^{\mathrm{perf}} \hookrightarrow \mathrm{Prism}$ *has a left adjoint given by:*

$$(A, I) \mapsto (A_{\mathrm{perf}}, IA_{\mathrm{perf}}) := ((\mathrm{colim}_\phi A)^\wedge_{p,I}, I(\mathrm{colim}_\phi A)^\wedge_{p,I}).$$

Proof. It suffices to show that the pair $((\mathrm{colim}_\phi A)^\wedge_{p,I}, I(\mathrm{colim}_\phi A)^\wedge_{p,I})$ is a prism. To see this, it remains to show that the ideal $I(\mathrm{colim}_\phi A)^\wedge_{p,I} \subseteq (\mathrm{colim}_\phi A)^\wedge_{p,I}$ is invertible. This follows form the fact that $(\mathrm{colim}_\phi A)^\wedge_{p,I}$ is a perfect δ-ring and Lemma **??**. □

2.4.2. *Perfectoid rings.* Next, we will see a relationship between perfectoid rings and perfect prisms.

Definition 2.32. A ring R is called a perfectoid ring if there is a perfect prism (A, I) and a ring isomorphism $R \cong A/I$. We denote by Perfd $\subseteq$ Ring the full subcategory of perfectoid rings.

Lemma 2.33. *Let R be a perfectoid ring. Then R is p-complete and $R[p] = R[p^2]$ holds.*

Proof. This follows from Lemma 2.31. □

Example 2.34. Any perfect $\mathbb{F}_p$-algebras are perfectoid. Indeed, Theorem 1.36 implies there is a canonical equivalence of categories:

$$\mathrm{Prism}^{\mathrm{perf}}_{(\mathbb{Z}_p,(p))/} \xrightarrow{\simeq} \mathrm{Ring}^{\mathrm{perf}}_{\mathbb{F}_p/} : (A, (p)) \mapsto A/p.$$

Conversely, any perfectoid ring of characteristic p is perfect.

We want to extend the above equivalence to the whole perfectoid rings and perfect prisms:

$$\mathrm{Prism}^{\mathrm{perf}} \xrightarrow{\simeq} \mathrm{Perfd} : (A, I) \mapsto A/I.$$

To do so, we will construct the quasi-inverse of $(A, I) \mapsto A/I$ using deformation theory.

Proposition 2.35. *The functor*

$$W(-) : \mathrm{Ring}^{\mathrm{perf}}_{\mathbb{F}_p/} \to \mathrm{Ring}^\wedge_p : R \mapsto W(R)$$

has a right adjoint

$$(-)^\flat : \mathrm{Ring}^\wedge_p \to \mathrm{Ring}^{\mathrm{perf}}_{\mathbb{F}_p/} : R \mapsto R^\flat := (R/p)^{\mathrm{perf}}.$$

We call the functor $(-)^\flat$ the tilting functor.

Proof. Let R be a perfect $\mathbb{F}_p$-algebra and A be a classically p-complete ring. Take an arbitrary ring map $R \to A/p$. Since $\mathbb{L}_{R/\mathbb{F}_p} \simeq 0$, deformation theory implies that there is a unique lift $W_n(R) \to A/p^n$ of the map $R \to A/p$. Passing to limit, we get a unique lift $W(R) \to A$. This implies the following natural bijections between mapping sets:

$$\begin{aligned}\mathrm{Hom}(R, A^\flat) &\simeq \mathrm{Hom}(R, A/p)\\ &\simeq \mathrm{Hom}(W(R), A).\end{aligned}$$

□

Definition 2.36. For any classically p-complete ring R, the counit map of the above adjunction is called the Fontaine's θ-map and denoted by

$$\theta : A_{\mathrm{inf}}(R) := W(R^\flat) \to R.$$

Theorem 2.37. *The functor*

$$\mathrm{Prism}^{\mathrm{perf}} \to \mathrm{Perfd} : (A, I) \mapsto A/I$$

is an equivalence of categories and having a quasi-inverse given by

$$R \mapsto (A_{\mathrm{inf}}(R), \ker\theta).$$

Proof. Let R be a perfectoid ring. We will prove that the pair $(A_{\mathrm{inf}}(R), \ker\theta)$ is actually a perfect prism. Let (A, I) be a perfect prism such that $R \cong A/I$. It suffices to show that there is an isomorphism of pairs $(A, I) \cong (A_{\mathrm{inf}}(R), \ker\theta)$. We have the following isomorphisms:

$$W(R^\flat)/p \cong R^\flat \cong \lim_\phi A/(I, p) \cong A/p,$$

where the last isomorphism comes from the fact that A/p is perfect. Since A and $W(R^\flat)$ are strict p-ring, this isomorphisms and Theorem 1.36 imply $W(R^\flat) \cong A$. Next, we will identify I and $\ker\theta$ under this isomorphism. Recall that the map $\theta : W(R^\flat) \to R$ is characterized as the unique lift of $R^\flat \to R/p$. On the other hand, the composition $A \to A/I \cong R$ is also a lift of this map. Thus, these two maps are identified under the isomorphism $A \cong A_{\mathrm{inf}}(R)$. In particular, we have $(A, I) \cong (A_{\mathrm{inf}}(R), \ker\theta)$. This argument also implies that θ is surjective, so $A_{\mathrm{inf}}(R)/\ker\theta \cong R$ holds. These arguments imply that the functor $R \mapsto (A_{\mathrm{inf}}(R), \ker\theta)$ is well defined and a quasi-inverse to the functor $(A, I) \mapsto A/I$. □

Theorem 2.38. *Let R be a ring. Then the following two assertions are equivalent:*

1. *R is perfectoid (i.e. there is a perfect prism (A, I) such that $A/I \cong R$),*
2. *there is an element $\pi \in R$ such that $\pi^p | p$, R is classically π-complete, R/p is semiperfect and the kernel of the map $\theta : A_{\mathrm{inf}}(R) \to R$ is principal.*

Proof. (1) ⇒ (2). It remains to prove that R is π-complete for some element $\pi \in R$ with $\pi^p|p$. Let $d \in A_{\mathrm{inf}}(R)$ be a distinguished element, then

$$\phi^{-1}(\xi)^p = \xi - p\phi^{-1}(\delta(\xi)).$$

Thus $\phi^{-1}(\xi)^p$ divides p in $A_{\mathrm{inf}}(R)/\xi = R$ (more precisely, $\phi^{-1}(\xi)^p = up$ for some unit $u \in A_{\mathrm{inf}}(R)$). Thus we get the conclusion.

(2) ⇒ (1). It suffices to show the following assertions:

- θ is surjective,
- $\ker\theta$ is generated by a distinguished element, say ξ,
- $A_{\mathrm{inf}}(R)$ is (p, ξ)-complete.

The surjectivity follows from the complete Nakayama's lemma and the fact that it is a lift of $R^\flat \to R/p$, which is surjective by our assumption.

Next, we will construct a generator ξ generating $\ker\theta$. Let u be an unit in R as in Lemma 2.39. Then we can define an element $p^\flat \in R^\flat$ such that $up = (p^\flat)^\sharp \in R$. Since θ is surjective, we can take a lift $v \in A_{\mathrm{inf}}(R)$ of u. Then the element $\xi := vp - [p^\flat]$ is contained in $\ker\theta$ by Lemma 2.41. Let d be a generator of $\ker\theta$. Then we can write as $\xi = dw$ with some element

$w \in A_{\mathrm{inf}}(R)$. We want to prove that w is invertible. This follows form Lemma 1.49. The derived (p, ξ)-completeness of $A_{\mathrm{inf}}(R)$ follows form Lemma 2.42. □

The following lemmas were used in the above proof:

Lemma 2.39. *Let R be a ring satisfying the condition (2) of Theorem* **??**. *Then there is a unit $u \in R$ such that $up \in R$ has a compatible system of p-power roots. Such a system defines an element*

$$p^\flat := ((up), (up)^{1/p}, (up)^{1/p^2}, \dots) \in \lim_\phi R/p = R^\flat.$$

Proof. First, we will prove that the map

$$R/\pi p \to R/\pi p : x \mapsto x^p$$

is a surjection of multiplicative monoids. Take an arbitrary element $x \in R$. Since R/π^p is semiperfect, we can write x as $x = y_0^p + \pi^p x_0$. By the same reason, we can write x_0 as $x_0 = y_1^p + \pi^p x_1$. Repeating this, we get:

$$x = \sum_{i=0}^{\infty} (y_i \pi^i)^p \equiv (\sum_{i=0}^{\infty} y_i \pi^i)^p \mod \pi p,$$

so we get the desired result.

Next, we prove the lemma using the result from the first paragraph. By the first paragraph, we can write $p = x_0^p + y_0 \pi p$, which implies $p(1 - y_0 \pi) = x_0^p$. Since $x_0^p | p$ and R is x_0-adically complete, the first paragraph also implies that the map $R/x_0 p \to R/x_0 p : a \mapsto a^p$ is also surjective. Thus we can write $x_0 = x_1^p + p x_0 y_1$, which implies $x_0(1 - p y_1) = x_1^p$. Repeating this arguments, we see that

$$p(1 - y_0 \pi) \prod_{i=0}^{\infty} (1 - y_i p)^{p^i}$$

has a compatible system of p-power roots of unity. □

Definition 2.40. Let R be a p-complete ring. Then we can easily check that the map $R^\flat \to R : (x_0, x_1, \dots) \mapsto \lim_{n \to \infty} (\widetilde{x}_i)^{p^i}$ is a well-defined map of multiplicative monoids. We call this map the $\sharp$-map and denote it by $x \mapsto x^\sharp$.

Lemma 2.41. *Let R be a p-complete ring. Then the Fontaine's θ-map $\theta : A_{\mathrm{inf}}(R) \to R$ can be written as*

$$\sum_{i=0}^{\infty} [a_i] p^i \mapsto \sum_{i=0}^{\infty} a_i^\sharp p^i.$$

Proof. It suffices to check $\theta([x]) = x^\sharp$ by the p-adic continuity of ring maps. By definition, we have the following commutative diagram:

$$\begin{array}{ccc} A_{\mathrm{inf}}(R) & \xrightarrow{\theta} & R \\ \downarrow & & \downarrow \\ R^\flat & \longrightarrow & R/p \end{array} .$$

Thus we have $\theta([x^{1/p^n}]) \equiv \widetilde{x}_n \mod p$, where $x = (x_0, x_1, \dots) \in \lim_\phi R \cong R^\flat$. Since θ is multiplicative, this implies $\theta([x]) \equiv (\widetilde{x}_n)^{p^n} \mod p^{n+1}$. Thus we get the desired result. □

Lemma 2.42. *Let R be a classically f-complete $\mathbb{F}_p$-algebra. Then $W(R)$ is classically $[f]$-complete.*

Proof. This follows form an easy direct computation. □

2.5. **Some Examples.**

2.6. **Animated prisms.**

3. Basic Properties of perfectoid rings

In § 2.4, we have seen a relation ship between perfect prisms and perfectoid rings. In this section, we will establish a basic properties and examples of perfectoid rings.

3.1. The tilting equivalence.

Theorem 3.1. *Let R be a perfectoid ring, $p^\flat \in R^\flat$ be an element such that $(p^\flat)^\sharp$ is equal to p times a unit. Then the tilting functor induces the following equivalence of categories:*

$$\mathrm{Perfd}_R \simeq (\mathrm{Perf}_{R^\flat})^\wedge_{p^\flat}.$$

Proof. Let $\xi \in A_{\mathrm{inf}}(R)$ be a generator of $\ker \theta_R$. Then $p^\flat R^\flat = A_{\mathrm{inf}}(R)/(\xi, p)$. We want to prove that the functor $S \mapsto W(S)/\xi$ is a quasi-inverse of the tilting functor.

Let R' be a perfectoid R-algebra. Then we have the following canonical isomorphisms:

$$W(R'^\flat)/\xi = W(R'^\flat)/\ker \theta_{R'} \cong R',$$

where the first equality follows form the rigidity of prismatic structures (Theorem 2.3). It is obvious form $p^\flat R^\flat = A_{\mathrm{inf}}(R)/(\xi, p)$ that $R'^\flat$ is $p^\flat$-complete.

Let S be a perfect $p^\flat$-complete $R^\flat$-algebra. Then we have the following isomorphisms:

$$(W(S)/\xi)^\flat \cong (W(S)/(\xi, p))^{\mathrm{perf}} \cong (S/p^\flat)^{\mathrm{perf}} \cong S,$$

where we used the perfectness of S in the last isomorphism.

Combining them, we get the conclusion. □

The above equivalence preserves the following ring-theoretic properties.

Proposition 3.2. *Let R be a perfectoid ring. Then R is a valuation ring if and only if $R^\flat$ is so. In this situation, we have $\mathrm{Frac}(R)^*/R^* \cong \mathrm{Frac}(R^\flat)^*/R^{\flat,*}$. In particular, the tilting functor preserves the rank of valuation rings.*

Proof. □

Proposition 3.3. *Let R be a perfectoid ring. Then R is an algebraically closed valuation ring if and only if $R^\flat$ is so.*

Proof. □

Proposition 3.4. *Let R be a perfectoid ring. $p^\flat \in R^\flat$ be an element such that $(p^\flat)^\sharp$ is p times a unit. Let $\pi^\flat \in R^\flat$ be an element such that $\pi^\flat | p^\flat$. Write $\pi := (\pi^\flat)^\sharp$. Then R is π-complete if and only if $R^\flat$ is $\pi^\flat$-complete.*

Proof. □

Proposition 3.5. *Let R be a π-complete perfectoid ring with $\pi^p | p$. Then there is an isomorphism $R[\pi] \cong R^\flat[\pi^\flat]$ of $A_{\mathrm{inf}}(R)$-modules. In particular, R is π-torsionfree if and only if $R^\flat$ is $\pi^\flat$-torsionfree.*

Proof. We first prove $R[p] \cong R^\flat[p^\flat]$. This follows form

$$R[p][1] \oplus R/p \simeq (A_{\mathrm{inf}}(R) \xrightarrow{\times p} A_{\mathrm{inf}}(R)) \otimes_{A_{\mathrm{inf}}(R)} (A_{\mathrm{inf}}(R) \xrightarrow{\xi} A_{\mathrm{inf}}(R)) \simeq R^\flat[p^\flat][1] \oplus R^\flat/p^\flat,$$

where $\xi \in A_{\mathrm{inf}}(R)$ is a distinguished element satisfying $\overline{\xi} = p^\flat \in R^\flat$.

Next, we prove the assertion in the proposition. Consider the following commutative diagram:

$$\begin{array}{ccc} A_{\mathrm{inf}}(R) & \xrightarrow{\theta} & R \\ \downarrow & & \downarrow \\ R^\flat & \longrightarrow & R/p \end{array}.$$

From this, we can write $\pi^\flat$ as $\pi^\flat = \widetilde{\pi} + a(\pi^\flat)^p$, where $\widetilde{\pi} \in A_{\mathrm{inf}}(R)$ is a lift of π along θ. Then the image $\overline{\widetilde{\pi}}$ of $\widetilde{\pi}$ in R equal to $\pi^\flat(1 - a(\pi^\flat)^{p-1})$, so

$$R[\pi] = (R[p])[\widetilde{\pi}] \cong (R^\flat[p^\flat])[\widetilde{\pi}] = R^\flat[\pi^\flat].$$

(One can directly check that this isomorphism comes from the$\sharp$-map). □

3.2. Reduction to p-torsionfree case.

Lemma 3.6. *Let R be a π-complete perfectoid ring with $\pi^p|p$. Assume that π has a compatible system of p-power roots of unity. Then the following hold:*

(1) $R[\pi] = R[\pi^{1/p^\infty}] = R[\pi^\infty]$,
(2) $R[\pi] \cap \pi^{1/p^\infty} R = \varnothing$.

Proof. (1). By Proposition **??**, we have

$$R[\pi] \cong R^\flat[\pi^\flat] = R^\flat[\pi^{\flat,1/p^\infty}] = R[\pi^{1/p^\infty}],$$

where we used the perfectness of $R^\flat$ at the second equality. The remaining part follows form an elementary calculation. (2). This follows directly form (1). □

Proposition 3.7. *Let R be a π-complete perfectoid ring with $\pi^p|p$. Then the quotient $\overline{R} := R/R[\pi]$ is a perfectoid π-torsionfree ring with tilt $\overline{R^\flat} := R^\flat/R^\flat[\pi^\flat]$.*

Proof. By the following exact sequence, $\overline{R^\flat}$ is $p^\flat$-complete.

$$0 \to R^\flat[p^\flat] \to R^\flat \to \overline{R^\flat} \to 0.$$

This implies that $\overline{R^\flat}$ is $p^\flat$-complete. Thus $W(\overline{R^\flat})/\xi$ is a perfectoid R-algebra, where $\xi \in A_{\mathrm{inf}}(R)$ is a distinguished element corresponding to R. Then we have the following commutative diagram:

$$\begin{array}{ccc} A_{\mathrm{inf}}(R) = W(R^\flat) & \twoheadrightarrow & W(\overline{R^\flat})/\xi \\ \downarrow & & \downarrow \\ R & \twoheadrightarrow & \overline{R} \end{array},$$

where the right vertical map is induced from the composition $A_{\mathrm{inf}}(R) \xrightarrow{\theta} R \twoheadrightarrow \overline{R}$ by the explicit description $\theta(\sum_{i=0}^\infty [a_i]p^i) = \sum_{i=0}^\infty a_i^\sharp p^i$ and the existence of an isomorphism $R^\flat[\pi^\flat] \underset{\cong}{\xrightarrow{\sharp}} R[\pi]$. On the other hand, since the perfectoid R-algebra $W(\overline{R^\flat})/\xi$ corresponds to the perfect $R^\flat$-algebra $\overline{R^\flat}$, Proposition **??** implies $W(\overline{R^\flat})/\xi$ is $\pi^\flat$-torsionfree. Thus the universality of $\overline{R}$ induces a unique map $\overline{R} \to W(\overline{R^\flat})/\xi$ fitting into the above diagram. Therefore, we see $W(\overline{R^\flat})/\xi \cong \overline{R}$ and $(\overline{R})^\flat \cong \overline{R^\flat}$. □

Theorem 3.8 (Factorization of perfectoid rings)**.** *Let R be a π-complete perfectoid ring with $\pi^p|p$. Then the canonical map*

$$R \to \overline{R} \times_{(\overline{R}/\pi)_{\mathrm{red}}} (R/\pi)_{\mathrm{red}}$$

is an isomorphism. In particular, any perfectoid ring can be written as a fiber product of π-torsionfree perfectoid ring and perfect ring.

Proof. The surjectivity follows form an easy diagram chasing, which works for arbitrary rings. The injectivity follows from Lemma **??**. The last assertion follows from the above proposition. □

Corollary 3.9. *Let R be a perfectoid ring. Then R is reduced.*

Proof. Let $\pi \in R$ be an element such that R is π-complete, $\pi^p|p$ and π has a compatible system of p-power roots of unity. By the above theorem, we may assume that R is π-torsionfree. Take any $x \in R$ such that $x^p = 0$. Since we have an isomorphism $R/\pi^{1/p} \xrightarrow{\cong} R/\pi : x \mapsto x^p$, we can write $x = \pi^{1/p}y$ using some $y \in R$. Then $0 = x^p = \pi y^p$. Since R is π-torsionfree, we get $y = 0$. In particular, $x = 0$. □

3.3. **p-integral closure.**

Proposition 3.10. *Let R be a π-complete π-torsionfree ring with $\pi^p|p$. Assume that the Frobenius map induces an isomorphism $R/\pi \xrightarrow{\cong} R/\pi^p$. Then R is a perfectoid ring.*

Proof. We want to use Theorem 2.38.

By replacing π with $u\pi$ using some unit $\pi \in R$, we may assume that π has a compatible system of p-power roots in R (see the proof of Lemma **??**). Then we can fined an element $\pi^\flat \in R^\flat$ such that $\pi^{\flat\sharp} = \pi$.

First, we will prove that the $\sharp$-map induce an isomorphism $R^\flat/\pi^\flat \xrightarrow{\cong} R/\pi$. By assumption, the ring R/π is semiperfect, so this map is surjective. For the injectivity, take an arbitrary element $x^\flat \in R^\flat$ such that $x^{\flat\sharp} \in \pi R$. Write $x^\flat = (x_0, x_1, \dots) \in \lim_\phi R \cong R^\flat$. Since $R/\pi \xrightarrow{(-)^p} R/\pi^p$ is bijective, $R/\pi^{1/p^n} \xrightarrow{(-)^{p^n}} R/\pi$ is also bijective. Thus we have $x_n \in \pi^{1/p^n}R$ for any n, which means $x^\flat \in \pi^\flat R$.

Next, let $\xi := p + x[\pi^\flat]^p$ be an element of $\ker\theta$ for some $x \in A_{\mathrm{inf}}(R)$. The composition

$$A_{\mathrm{inf}}(R) \xrightarrow{\theta} R \to R/\pi$$

coincides with

$$A_{\mathrm{inf}}(R) \to A_{\mathrm{inf}}(R)/(\xi, [\pi^\flat]) \cong R^\flat/\pi^\flat \underset{\cong}{\xrightarrow{\sharp}} R/\pi.$$

Thus, we have

$$\ker\theta \subseteq (\xi, [\pi^\flat]).$$

Therefore, any element $y \in \ker\theta$ can be written as $y = a_0\xi + b_0[\pi^\flat]$ using some elements $a_0, b_0 \in A_{\mathrm{inf}}(R)$. Since R is π-torsionfree, the following equality implies $b \in \ker\theta$:

$$0 = \theta(y) = \pi\theta(b_0).$$

Thus we can write b_0 as $b_0 = a_1\xi + b_1[\pi^\flat]$. Repeating this procedure, we can write y as the following:

$$y = \xi(b_0 + [\pi^\flat]b_1 + [\pi^\flat]^2 b_2 + \cdots) \in \xi A_{\mathrm{inf}}(R).$$

Therefore, we have $\ker\theta = \xi A_{\mathrm{inf}}(R)$. □

Definition 3.11. An extension of rings $A \subseteq B$ is called p-integrally closed if $x \in A$ holds for any element $x \in B$ with $x^p \in A$.

Lemma 3.12. *Let S be a ring, $\pi \in S$ be a non-zerodivisor with $\pi^p|p \in S$. then the followings are equivalent:*

(1) *the p-power map $S/\pi \to S/\pi^p$ is injective,*
(2) *the inclusion $S \subseteq S[1/\pi]$ is p-integrally closed.*

Proof. (1) ⇒ (2). Take an element $b = a/\pi^m \in S[1/\pi]$ such that $a \in S$ and $b^p \in S$. Then we have $a^p \in \pi^{pm}S$. By (1), we see that there exists an element $c \in S$ such that $a = \pi c$. This implies that $c^p \in \pi^{(m-1)p}S$ by π-torsionfree-ness of S. Applying this argument to c in place of a, we have $c \in \pi S$, thus $a \in \pi^2 S$. Repeating this procedure, we have $a \in \pi^m S$, that is, $b \in S$.

(2) ⇒ (1). Let $a \in S$ be an element such that $a^p \in \pi^p S$. This means that $(a/\pi)^p \in S$. By (2), we have $a/\pi \in S$. □

Proposition 3.13. *Let R be a π-complete π-torsionfree ring with $\pi^p|p$. Let S be the p-integral closure of R in $R[1/\pi]$. Then $S_p^\wedge$ is a perfectoid ring.*

Proof. By Proposition 3.10, it suffices to show that the Frobenius map induces an isomorphism $S/\pi \xrightarrow[\cong]{(-)^p} S/\pi^p$. To prove this, we will fined an explicit description of the p-integral closure S.

For each $a \in R$ with $a^p \in \pi^p R$, take a sequence $\{a_n\}_{n\in\mathbb{N}}$ such that $a_0 = a$ and $a_{n+1}^p \equiv a_n \mod \pi^p$ holds for all $n \geq 0$. (Such a sequence exists by the assumption of the proposition). One can easily check that $a_n/\pi^{1/p^n} \in S$ holds. Thus, we can define a subring of S as follows:

$$R_1 := R[a_n/\pi^{1/p^n} | n \geq 0, a \in R \text{ s.t. } a^p \in \pi^p R].$$

Then one can directly check that this ring also satisfies the condition of the proposition, so we can construct a subring R_2 of S by the same way using R_1 instead of R. Repeating this procedure, we have a sequence of subrings $\{R_n\}_{n\in\mathbb{N}}$ of S. Let $R_\infty := \cup_n R_n \subseteq S$. By construction, the p-power map $R_\infty/\pi \to R_\infty/\pi^p$ is isomorphic. Thus R_∞ is also p-integrally closed, so $R_\infty = S$ holds. Moreover, the π-completion of R_∞ is perfectoid. □

3.4. **Some examples.**

Example 3.14.
(1) $\mathbb{Z}_p^{\mathrm{cyc}} := \mathbb{Z}_p[\zeta_{p^\infty}]_p^\wedge$ is a perfectoid ring. This follows from a direct computation or from Proposition 3.13.
(2) $\mathbb{Z}_p[p^{1/p^\infty}]_p^\wedge$ is a perfectoid ring. This follows from a direct computation or from Proposition 3.13.
(3) One can easily check that $\mathbb{Z}_p^{\mathrm{cyc},\flat} \cong (\mathbb{Z}_p[p^{1/p^\infty}]_p^\wedge)^\flat$. This is an example showing that the tilting functor is not fully faithful even if the domain is restricted to mixed characteristic perfectoid rings.
(4) Let C be a algebraically closed complete valuation field of height one. Then the ring $\mathcal{O}_C$ of integers in C is perfectoid. This follows form Proposition 3.13.

Next, we will learn how to construct new perfectoid rings from given ones.

3.4.1. *Stability under certain limits.*

Proposition 3.15. *Let $R \to T \leftarrow S$ be a diagram of perfectoid rings. Assume that $R \to T$ is surjective. Then the fiber product $R \times_T S$ is also perfectoid with tilt $R^\flat \times_{T^\flat} S^\flat$.*

Proof. Write $Q := R \times_T S$. Since the tilting functor

$$(-)^\flat : \mathrm{Ring}_p^\wedge \to \mathrm{Ring}_{\mathbb{F}_p}^{\mathrm{perf}}$$

has a left adjoint $W(-)$, it preserves all limits. From this and the fact that the functor $W(-)$ preserves all limits, which follows form a direct computation or the existence of a left adjoint (Theorem 1.6), we see that the functor $A_{\mathrm{inf}}(-)$ also preserves all limits. Thus we get the following pullback diagram:

$$\begin{array}{ccc} A_{\mathrm{inf}}(Q) & \longrightarrow & A_{\mathrm{inf}}(R) \\ \downarrow & & \downarrow \\ A_{\mathrm{inf}}(S) & \longrightarrow & A_{\mathrm{inf}}(T) \end{array}.$$

Since $R \to T$ is surjective, $A_{\mathrm{inf}}(R) \to A_{\mathrm{inf}}(T)$ is also surjective. Thus we get the following fiber sequence:

$$A_{\mathrm{inf}}(Q) \xrightarrow{a\mapsto(a,a)} A_{\mathrm{inf}}(R) \oplus A_{\mathrm{inf}}(S) \xrightarrow{(a,b)\mapsto a-b} A_{\mathrm{inf}}(T) \to .$$

On the other hand, let $\xi_S \in A_{\mathrm{inf}}(S)$ (resp. $\xi_R \in A_{\mathrm{inf}}(R)$) be a distinguished element corresponding to S (resp. R). Then $\xi_R = u\xi_S \in A_{\mathrm{inf}}(T)$ using some unit $u \in A_{\mathrm{inf}}(T)$. Let $v \in A_{\mathrm{inf}}(R)$ be a lift of u. Then v is also a unit element (because unit elements in $A_{\mathrm{inf}}(R) = W(R^\flat)$ coincides with elements $(x_0, x_1, \dots) \in W(R^\flat)$ such that $x_0 \neq 0$). Since $v\xi_R \in A_{\mathrm{inf}}(R)$ is also distinguished, we can assume $\xi_R = \xi_S \in A_{\mathrm{inf}}(T)$ by replacing ξ_R with $v\xi_R$. Then the element $\xi := (\xi_R, \xi_S) \in A_{\mathrm{inf}}(R) \times_{A_{\mathrm{inf}}(T)} A_{\mathrm{inf}}(S) \cong A_{\mathrm{inf}}(Q)$ is distinguished (because $\delta((\xi_R, \xi_S)) = (\delta(\xi_R), \delta(\xi_S))$).

Next, consider the derived quotient of the above fiber sequence by ξ:

$$A_{\mathrm{inf}}(Q)/\xi \to R \oplus S \to T \to .$$

From this, we see that the perfectoid ring $A_{\mathrm{inf}}(Q)/\xi$ coincides with

$$Q = \mathrm{Fib}(R \oplus S \to T).$$

Therefore, Q is also a perfectoid ring with tilt $R^\flat \times_{T^\flat} S^\flat$. □

Remark 3.16. In the above proposition, the surjectivity assumption is necessary. For example, consider the diagram of perfectoid rings

$$\mathbb{Z}_p[p^{1/p^\infty}]_p^\wedge \hookrightarrow \mathcal{O}_C \hookleftarrow \mathbb{Z}_p[\zeta_{p^\infty}]_p^\wedge,$$

where p is an odd prime number and $\mathcal{O}_C$ is the ring of integers of the p-adic complex number field $C := \widehat{\overline{\mathbb{Q}}}_p$. To prove that the fiber product

$$\mathcal{O}_K := \mathbb{Z}_p[p^{1/p^\infty}]_p^\wedge \times_{\mathcal{O}_C} \mathbb{Z}_p[\zeta_{p^\infty}]_p^\wedge = \mathbb{Z}_p[p^{1/p^\infty}]_p^\wedge \cap \mathbb{Z}_p[\zeta_{p^\infty}]_p^\wedge$$

is not perfectoid, it suffices to show

$$K_n := (\mathbb{Q}_p(p^{1/p^n}) \cap \mathbb{Q}_p(\zeta_{p^n})) = \mathbb{Q}_p$$

for any $n \geq 1$. By the Kummer theory, we see

$$[\mathbb{Q}_p(\zeta_{p^n}, p^{1/p^n}) : \mathbb{Q}_p(\zeta_{p^n})] = p^n.$$

On the other hand, since $X^p - p \in \mathbb{Q}_p[X]$ is an Eisenstein polynomial, we see

$$[\mathbb{Q}_p(p^{1/p^n}) : \mathbb{Q}_p] = p^n.$$

Combining them, we get the desired result. This example also implies that the category Perfd of perfectoid rings does not admit arbitrary fiber products.

Proposition 3.17. *Let $\{R_i\}_{i\in I}$ be a set of perfectoid rings. Them the product ring $\prod_{i\in I} R_i$ is also perfectoid and it tilt is $\prod_{i\in I} R_i^\flat$.*

Proof. Let $\xi_i \in W(R_i^\flat)$ be a distinguished element corresponding R_i for each $i \in I$. Then

$$\xi := (\xi_i) \in \prod_{i\in I} W(R_i^\flat) \cong W(\prod_{i\in I} R_i^\flat)$$

is distinguished and

$$(\prod_{i\in I} W(R_i^\flat))/\xi \cong \prod_{i\in I} W(R_i^\flat)/\xi_i \cong \prod_{i\in I} R_i.$$

Therefore, $\prod_{i\in I} R_i$ is a perfectoid ring whose tilt is $\prod_{i\in I} R_i^\flat$. □

3.4.2. *Stability under certain colimits.*

Proposition 3.18. *Let $\{R_i\}_{i\in I}$ be a filtered diagram of perfectoid rings. Then $(\mathrm{colim}_{i\in I} R_i)_p^\wedge$ is also a perfectoid ring, whose tilt is $(\mathrm{colim}_{i\in I} R_i^\flat)_{p^\flat}^\wedge$.*

Proof. We may assume I has an initial object i_0. Let $\xi \in W(R_{i_0}^\flat)$ be a distinguished element corresponding to R_{i_0}. Then the pair $((\mathrm{colim}_{i\in I} W(R_i^\flat))_{(p,\xi)}^\wedge, (\xi))$ is a perfect prism, whose corresponding perfectoid ring is $(\mathrm{colim}_{i\in I} R_i)_p^\wedge$. Since we have

$$(\mathrm{colim}_{i\in I} W(R_i^\flat))_{(p,\xi)}^\wedge/p \cong (\mathrm{colim}_{i\in I} R_i^\flat)_{p^\flat}^\wedge,$$

the tilt of the perfectoid ring $(\mathrm{colim}_{i\in I} R_i)_p^\wedge$ is $(\mathrm{colim}_{i\in I} R_i^\flat)_{p^\flat}^\wedge$. □

Proposition 3.19. *Let $R \leftarrow T \to S$ be a diagram of perfectoid rings. Then $(R \otimes_T S)_p^\wedge$ is also a perfectoid ring, whose tilt is $(R^\flat \otimes_{T^\flat} S^\flat)_{p^\flat}^\wedge$.*

Proof. This follows from the same proof of the above proposition. □

Remark 3.20. The inclusion Perfd $\hookrightarrow \mathrm{Ring}_p^\wedge$ does not preserve coproducts. Indeed, the tensor product $R \otimes_{\mathbb{Z}_p} \mathbb{F}_p \cong R/p$ does not need to be perfect. Indeed, the correct coproduct of R and $\mathbb{F}_p$ in Perfd is the perfect closure of R/p. More generally, for any perfectoid rings R and S, their coproduct in Perfd is the perfectoidization of $R \otimes_{\mathbb{Z}_p} S$. The notion of perfectoidization will be treated in §**??**

3.4.3. *Stability under other operations.*

Proposition 3.21. *Let R be a perfectoid ring. Then $R[X^{1/p^\infty}]_p^\wedge$ is also perfectoid, whose tilt is $R^\flat[X^{1/p^\infty}]_{p^\flat}^\wedge$.*

To prove this, we prepare the next lemma:

Lemma 3.22. *Let S be a semiperfect $\mathbb{F}_p$-algebra. Then the map*

$$f : W_n(S)[X^{1/p^\infty}] \xrightarrow{X^{1/p^n} \mapsto [X^{1/p^\infty}]} W_n(S[X^{1/p^\infty}])$$

is an isomorphism.

Proof. To prove the surjectivity, we may assume $S = \mathbb{F}_p[Y_i^{1/p^\infty} | i \in I]$, because there is a surjection from this ring to S. In this case, the surjectivity follows form the following isomorphism:

$$W_n(\mathbb{F}_p[Y_i^{1/p^\infty} | i \in I])[X^{1/p^\infty}] \cong \mathbb{Z}/p^n[X^{1/p^\infty}, Y_i^{1/p^\infty} | i \in I] \cong W_n(\mathbb{F}_p[X^{1/p^\infty}, Y_i^{1/p^\infty} | i \in I]).$$

Next, we prove the injectivity. Let $\sum_{i=0}^m (a_{0,i}, \dots, a_{n-1,i}) X^{l_i}$ be an arbitrary element of the domain, where $l_i \in \mathbb{Z}[1/p] \cap \mathbb{Q}_{>0}$ with $i \neq j \Rightarrow l_i \neq l_j$. Assume $f(\sum_{i=0}^m (a_{0,i}, \dots, a_{n-1,i}) X^{l_i}) = 0$. The first coordinate of $f(\sum_{i=0}^m (a_{0,i}, \dots, a_{n-1,i}) X^{l_i}) = 0$ is $\sum_{i=0}^m a_{0,i} X^{l_i} = 0$, so we have $a_{0,i} = 0$ for all i. Using this result, we see that the second coordinate of $f(\sum_{i=0}^m (a_{0,i}, \dots, a_{n-1,i}) X^{l_i})$ is $\sum_{i=0}^m a_{1,i} X_i^{pl_i} = 0$, which implies $a_{1,i} = 0$ for any i. Repeating similar arguments, we get the conclusion. □

Proof of the proposition. The composition

$$R \cong W(R^\flat)/\xi \to W(R^\flat[X^{1/p^\infty}]_{p^\flat}^\wedge)/\xi$$

induces a map

$$R[X^{1/p^\infty}]_p^\wedge \to W(R^\flat[X^{1/p^\infty}]_{p^\flat}^\wedge)/\xi,$$

where ξ is a distinguished element of $W(R^\flat)$ corresponding to R. It suffices to show that this map is an isomorphism. Since we have $R \cong W(R^\flat)/\xi$, it reduces to show that the map

$$W_n(R^\flat)[X^{1/p^\infty}] \to W_n(R^\flat[X^{1/p^\infty}])$$

is an isomorphism for any n. This is a special case of the above lemma. □

Proposition 3.23. *Let R be a perfectoid ring. $f^\flat \in R^\flat$ be an element. Write $f := f^{\flat\sharp} \in R$. Then the derived f-completion $R_f^\wedge$ coincides with the classical f-completion and is a perfectoid ring. The tilt of $R_f^\wedge$ is $(R^\flat)_{f^\flat}^\wedge$.*

Proof. Since R is reduced (Corollary 3.9), it has bounded f-power torsion. Thus its derived f-completion coincides with classical f-completion.

Next, we will prove the remaining part. Since $(R^\flat)_{f^\flat}^\wedge$ is $f^\flat$-complete, the ring $W((R^\flat)_{f^\flat}^\wedge)/\xi$ is f-complete, so we have a natural map $R_f^\wedge \to W((R^\flat)_{f^\flat}^\wedge)/\xi$. We want to prove that this map is an isomorphism. To see this, it suffices to show the bijectivity after taking modulo f^n, which it straightforward. □

References

[BMS18] B. Bhatt, M. Morrow, P. Scholze. *Integral p-adic Hodge theory.* Publ. Math. Inst. Hautes Études Sci. 128 (2018), 219–397. MR 3905467

[BS22] B. Bhatt, P. Scholze. *Prisms and prismatic cohomology.* Ann. of Math. (2) 196 (2022), no. 3, 1135–1275. MR4502597

[CS24] K. Česnavičius and P. Scholze. *Purity for flat cohomology.* Ann. of Math. (2) **199** (2024), no. 1, 51–180; MR4681144.

[Rez19] C. Rezk. *Etale extensions of λ-rings.* `https://rezk.web.illinois.edu/etale-lambda.pdf`.

[Ser79] J.P Serre. *Local fields.* translated from the French by Marvin Jay Greenberg, Graduate Texts in Mathematics, 67, Springer, New York-Berlin, 1979; MR0554237.

[Sta26] Stacks Project Authors. *Stacks Project.*